Coarse Grained Variables and Deterministic Chaos in an Excitable System

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Temporal coarse graining was applied to the dynamical variables of a semiconductor laser with optical feedback. The chaotic low frequency fluctuations obtained in numerical and experimental data are shown to have properties of a self-excitable deterministic system. External exciting noise is replaced by the ultrafast chaotic oscillations of the system. A low dimensional coarse-grained phase space is defined and time constants are introduced and measured for the exponential drop and recovery of the randomly excited equally shaped spikes.

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Excitability in a dynamical system is defined when the occurrence of large pulses can be triggered by small perturbations. These large pulses, or excited spikes, have a proper shape, usually dictated by a low dimensional phase space. An excitable system may exhibit stochastic behavior when the triggering perturbation issues from a noise source. Excitability was identified in many natural systems, including lasers [1–3].

Considering that deterministic fluctuations with properties of noise are also well known in nonlinear dynamics [4–6], one can conceive chaotic deterministic self-excitable dynamical systems. They result from high dimensional deterministic chaotic dynamics, when observed in a coarse-grained time scale. At this rough scale, equally shaped spikes may emerge similar to the ones of an externally excitable system. So, what is usually described as an oscillatory behavior may be interpreted as self-excitation of an excitable system. Such is the subject of this Letter.

Ultrafast pulsations or oscillations of the original dynamical variables are cast in the form of ''pseudostochastic'' noise. New coarse-grained variables can be defined, averaging to smooth the ultrafast scale, thus forming a low dimensional system. These new variables live in a phase space having slower dynamics, organized by different fixed points and cycles. The parameters of the original high dimensional system simultaneously determine the slow system phase space and the noise-equivalent fast fluctuations.

A simplified two dimensional model for an excitable system is in Fig. 1. For well chosen parameters, a stable node N exists near a saddle point S, and they are connected by a heteroclinic orbit. In the connection, the stable manifold of N has a negative eigenvalue, which is the inverse of the exponential recovery time constant, τ_r, of the trajectories when they are nearby and approaching N. Conversely, the unstable manifold of S will have a positive eigenvalue, inverse of the time constant τ_d, of trajectories escaping from the neighborhood of S. Excitability results from fluctuations that make the system jump from the stable neighborhood of N across the separatrix stable manifold of S and into a cycle guided by the connected manifolds. For parameters giving excitable spikes, a time T, much longer than the two time constants, will separate consecutive pulses. Each one will depart from the neighborhood of N with the time constant τ_d and recover to the point N with a time constant τ_r.

In the following we describe calculations and experiments on the temporally coarse-grained variables of a diode laser with optical feedback. Low frequency fluctuations (LFF) are interpreted as excited spikes of a dynamical system, as proposed by Eguia and Mindlin [2] and schematized in Fig. 1. Herein, the numerical solution of the laser equations are done without stochastic terms, and so we verify that the system behaves as a deterministic chaotic self-excitable system [7]. The relation between excitability and self-pulsations in semiconductor lasers have been previously studied, beyond the context of LFF, by Krauskopf *et al.* [8].

Low frequency fluctuations in the power output of diode lasers with optical feedback has been reported nearly three decades ago [9]. The feedback has a time delay τ in the range of tens of ns when a reflecting mirror is located a few meters from the laser, creating a so-called external optical cavity. Along with the slow LFF events, the laser dynamics has well known very fast (tens to hundreds of ps range) pulsations in its power output [10–16]. It has also been known for a long time that the experimental power output, when detected with an electronic low-pass filter having a time constant of nanoseconds, gives the typical LFF events having the shape of a sharp drop followed by a stepwise

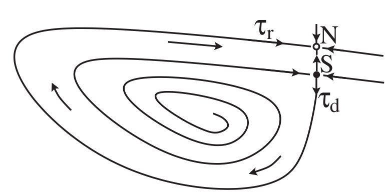

FIG. 1. Scheme of fixed points and manifolds for a bidimensional excitable dynamical system as proposed in [2].

recovery [9,17,18]. The deterministic and random contributions to the origin of the LFF drops are still the subject of studies, and many works have been dedicated to the measurement, calculation, and interpretation of the properties of T [19–23]. As for the drop and recovery process, few measurements exist. It was established that within long time series, with large spread values of T, the recovery of the stepwise drops is always nearly the same [1,17,24]. Hegarty *et al.* [18] have also reported constancy for the fast population relaxation oscillations in these systems. The study of the semiconductor laser with optical feedback as an excitable dynamical system, driven by noise in its pump current, shows excited power drop spikes having the same shape as the LFF events [1,2,19,25].

Let us begin with the numerical model. The equations for the laser are [10]

$$\dot{E} = \frac{(1+i\alpha)}{2}\left[G(N) - \frac{1}{\tau_p}\right]E(t) + \kappa E(t-\tau)e^{i\omega_0\tau}$$
$$\dot{N} = J - N(t)/\tau_s - G(N)|E(t)|^2, \tag{1}$$

where the laser gain is given by

$$G(N) = G_0(N - N_0)(1 + \epsilon|E(t)|^2)^{-1}. \tag{2}$$

The various parameters and their typical values are well discussed in the literature [7,12]. With a fourth order Runge-Kutta algorithm one obtains numerical chaotic behavior for $E(t)$ and $N(t)$. The time scales in the integrations were fixed by $dt = 1$ ps, $\tau_p \sim 5$ ps, $\kappa \sim 0$–$20\ \mathrm{ns}^{-1}$, $\tau_s \sim 1$ ns, and $\tau \sim 10$ ns. Segments of the calculated laser intensity, $I(t) = |E(t)|^2$, are shown in Fig. 2. Without feedback, the laser operates with a constant intensity. This is the stable node dynamical condition. Ultrafast chaotic spikes of the order of hundreds of picoseconds appear when there is optical feedback (Fig. 2). Small values of κ reveal a fast fluctuating intensity. However, the well known low frequency fluctuations do not appear in Fig. 2(a). The system is below the threshold needed to self-excite the LFF power drops. Increasing the feedback sets in the excitation of LFF. This is shown in Fig. 2(b). A coarse-grained averaged power signal, $P(t)$, is also shown. It is calculated as

$$P(t) = \frac{1}{\tau_f}\int_{-\infty}^{t} \exp[-(t-t')/\tau_f]I(t')dt', \tag{3}$$

where τ_f is equivalent to the optical detection filter time constant, taken as 1 ns. The time T between LFF drops has a well studied random distribution. Its irregularity was created by the deterministic fast fluctuations for there is no stochastic term in the equations.

One new result to be emphasized here is that the coarse graining, used in many previous works, is not just a convenient time averaging procedure to visualize the LFF drops. Only with such time coarse-grained variables can one refer to the excitable events having a fall and recovery time, as shown in Fig. 3. The existence of the range of values for the feedback rate where the noiselike fast fluctuations appear—but its value is not enough to excite the large spikes of LFF [Fig. 2(a)]—allows us to argue for a threshold in LFF self-excitability with parameter variation. Unfortunately, the system has too many parameters controlling its dynamics and a sharp threshold did not appear just by variation of κ. Notice that the maximum amplitude of the fast power fluctuation in Fig. 3 occurs much earlier than the LFF drop. Thus, the equivalent noise exciting LFF is due not only to the amplitude of the power fluctuation. The field phase fluctuations, which we do not show herein, contribute to the process. Such contribution is related to the quasimode locking process that creates the ultrashort pulses [26].

The LFF drops, once excited, always have almost the same coarse-grained shape. Described as very fast by previous authors concerned with the statistics of the time T between drops, the fall and recovery have a specific shape, experimentally studied in [24]. Within our interpretation, the early stage of the drops and the recoveries have coarse-grained exponential time dependences, with respective time constants τ_d and τ_r. These can be measured and

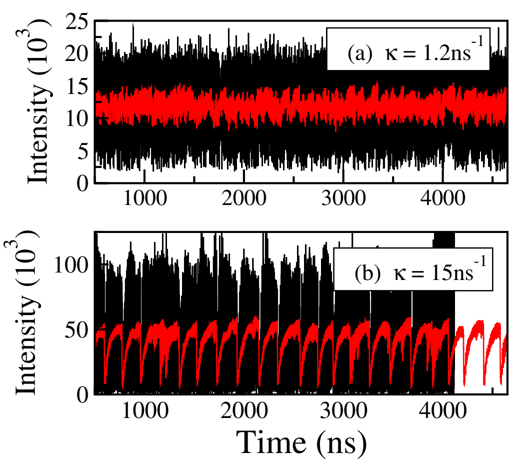

FIG. 2 (color online). Calculated time series for the laser power: (a) with weak feedback, $\kappa = 1.2\ \mathrm{ns}^{-1}$, chaos appears as fast pulsation, in black, but there is no LFF. (b) Stronger feedback, $\kappa = 15\ \mathrm{ns}^{-1}$, puts the system above threshold for excitation of LFF events. In gray (color) are the coarse-grained series.

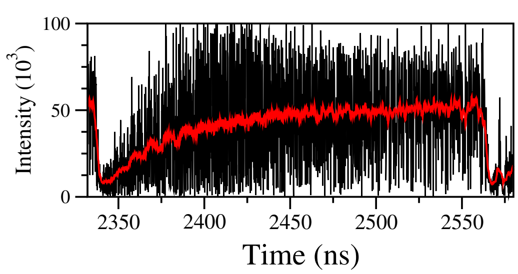

FIG. 3 (color online). One LFF drop and recovery from Fig. 2(b).

must be accounted for in any understanding of the physics of the laser. Numerically and experimentally the drops here are typically very fast and τ_d is more than an order of magnitude smaller than τ_r when the laser has round trip feedback time in the range of 3 to 60 ns.

To inspect the attractor of the chaotic motion, a phase space projection of the calculated laser power versus the carrier number is plotted in Fig. 4, for conditions where LFF spikes occur. With the calculated ultrafast variables, represented with a black line, the result is a densely visited area of the projection plane. Such is a typical phase space projection plane for a very high dimensional dynamical system. The temporally coarse-grained variables, on the other hand, reveals a simplified picture. Once the filtering time exceeds 1 ns, an almost functional dependence, $\overline{N(t)} \sim P(t)^{-1}$, results. This is the squeezed cycle shown in gray (red) in Fig. 4. Recently, similar calculations of power and carrier number for a semiconductor laser with LFF have been done and confirmed experimentally [27]. For comparison with experiments, another phase portrait, the calculated laser power versus its time derivative, is given in Fig. 5. Again, only with the aid of temporal coarse-grained variables in the new phase space, one sees the characteristic behavior of an excitable system. The time evolution, running clockwise in the figure, corresponds to cycles occurring after long and irregular intervals of duration T where the average laser power is nearly constant and maximum. Those intervals are represented by the heavily visited region of the upper right corner of Fig. 5(b) and would correspond to the neighborhood of point N in Fig. 1. Each self-excited LFF cycle initiates with a straight drop with sharp negative slope. This must correspond to an exponential escape and so we take $\dot{P}/P = -1/\tau_d$ in this region. After some decay, $P(t)$ loses its exponential temporal evolution entering the deep nonlinear behavior distant from the fixed points. Once passed by its minimum, in the left upper portion of the figure, the coarse-grained power begins a recovery, again on a straight line (i.e., exponential time dependence) from left to right on the upper part of the figure, where $\dot{P}/P = 1/\tau_r$. From the data of Fig. 5 we obtain $\tau_r \approx 40$ ns and $\tau_d \leq 2$ ns while from the average on the time series $\bar{T} \approx 200$ ns.

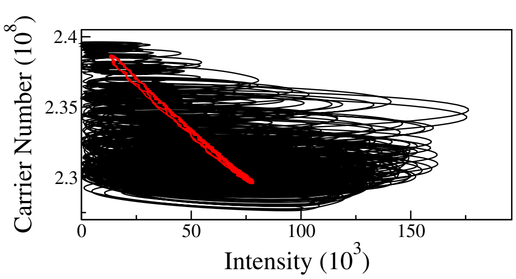

FIG. 4 (color online). Phase portrait: Laser intensity versus the carrier number, with ps time resolution (black), and after coarse graining (red online).

Experimental measurements were made in an SDL 5401 GaAlAs semiconductor laser emitting at 850 nm. It was thermally stabilized to 0.01 K and had solitary threshold current of 17 mA. An external flat mirror with distance fixed between 0.9 m to 9.0 m was used to create the optical feedback. Collimating lenses controlled the amount of feedback, measured by the threshold reduction parameter [12]. The intensity output was detected by a 1.5 GHz bandwidth photodiode, and the best data series were acquired by a digital oscilloscope having a bandwidth of 300 MHz and a maximum sampling rate of 2.5 GS/s. A typical experimental segment showing LFF drops is seen in Fig. 6(a).

Long data series were stored in a 12 bits A/D acquisition system running at 100 MHz. A phase portrait of the power versus its derivative for a typical series with 10^4 LFF drops is shown in Fig. 6(b). The experimental drops of the LFF events could barely be resolved by our acquisition system. Yet, from the data we infer a fall time constant, τ_d, of 2 ns for the laser with feedback delay time between 6 and 60 ns. This nearly constant value of τ_d was also verified numerically. It is important to emphasize that the drops do not occur in a time scale shorter than 1 ns. They have many fast oscillations within their occurrence and so can be defined only for the variables of the coarse-grained system.

The recovery time constant, τ_r, is also to be defined as a property of the coarse-grained system. Its dependence with pump current variation was recently measured [24].

To summarize we have experimentally studied the time filtered chaotic LFF pulses of a diode laser with optical feedback and calculated temporally coarse-grained variables from the Lang-Kobayashi equations describing these lasers. These coarse grain variables are the ones for the description of the system as a deterministic excitable dy-

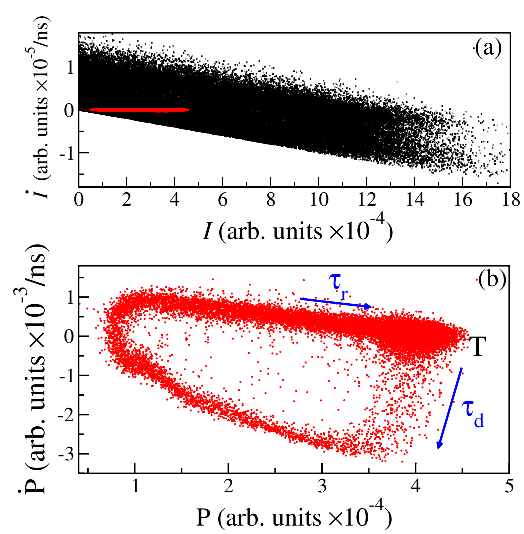

FIG. 5 (color online). Phase portraits of the laser intensity versus its derivative. (a) Dark dots with the original ps time resolution. (b) An expanded scale of the coarse-grained portrait.

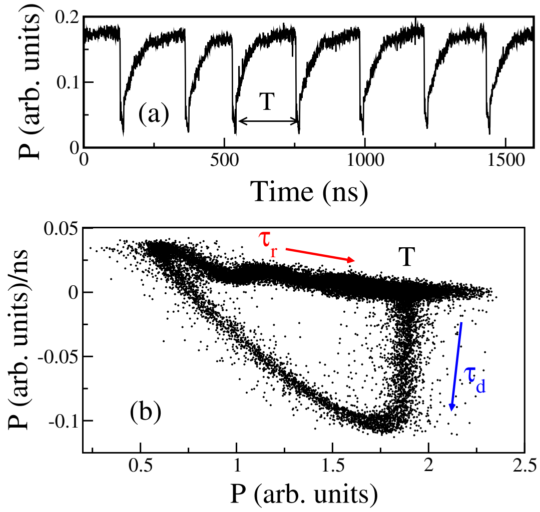

FIG. 6 (color online). (a) Segment of experimental time series. (b) Phase portrait of the experimental laser intensity with LFF.

namical system. The ultrafast fluctuations in the dynamics play the role of a noise source [7] that excites the LFF cycles, and so a slow time scale behavior, driven by stochastic fluctuations, is predicted from deterministic equations. Time constants for the LFF events, one for the drop and another for the recovery were introduced and measured. The value for the drop time constant was too fast for a precise measurement, but its physical characterization as a property of the coarse-grained picture is clearly evidenced. A conceptually relevant recent work by Torcini *et al.* [28] presents LFF as transient when calculated with the Lang and Kobayashi [10] equations without noise. They show that these transients are of very long time duration, orders of magnitude longer than the times used to observe (numerically and experimentally) the properties of LFF discussed here. This is totally compatible with our view that once started, the LFF burst has its properties controlled by the classical deterministic fast pulsations of the electromagnetic laser field. These are larger than spontaneous emission noise when the laser is on. The view of LFF as self-excited pulses of an excitable system, resulting from deterministic fast pulsations, as given here, may have impact on the interpretation of recent results on laser synchronization [29]. It is also natural to foresee the extension of the concept of deterministic self-excitable systems beyond the domain of physics.

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