Self-similarity, symmetries and asymptotic behavior in Morrey spaces for a fractional wave equation

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Abstract

This paper is concerned with a fractional PDE that interpolates semilinear heat and wave equations. We show results on global-in-time well-posedness for small initial data in the critical Morrey spaces and space dimension $n \geq 1$. We also remark how to derive the local-in-time version of the results. Qualitative properties of solutions like self-similarity, antisymmetry and positivity are also investigated. Moreover, we analyze the asymptotic stability of the solutions and obtain a class of asymptotically self-similar solutions.

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1 Introduction

Differential equations of fractional order appear naturally in several fields such as physics, chemistry and engineering by modelling phenomena in viscoelasticity, thermoelasticity, processes in media with fractal geometry, heat flow in material with memory and many others. The two most common types of fractional derivatives acting on time variable t are those of Riemann-Liouville and Caputo.We refer the surveys [21, 22], [33] and [29] in which the reader can find a good bibliography for applications on those fields.

Here we are interested in a semilinear integro-partial differential equation, which interpolates the semilinear heat and wave equations, and reads as

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\Delta_x u(s) + |u(s)|^{\rho} u(s)) ds, \text{ in } x \in \mathbb{R}^n, \ t > 0, \quad (1.1)$$

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where $n \geq 1$, $1 < \alpha < 2$, $0 < \rho < \infty$, $\Gamma(\alpha)$ denotes the Gamma function and Δ_x is the laplacian in the x-variable. This equation is formally equivalent to the IVP for the fractional partial differential equation (FPDE)

$$\partial_t^\alpha u = \Delta_x u + |u|^\rho u, \quad \text{in } \mathbb{R}^n, \ t > 0, \tag{1.2}$$

$$u(x,0) = u_0 \text{ and } \partial_t u(x,0) = 0 \text{ in } \mathbb{R}^n, \tag{1.3}$$

where $\partial_t^\alpha u = \mathbf{D}_t^{\alpha-1}(\partial_t u)$ and $\mathbf{D}_t^{\alpha-1}$ stands for the Riemann-Liouville derivative of order $\alpha - 1$. For a Lebesgue measurable function f and $k = \lfloor \eta \rfloor + 1$ with $\eta \geq 0$ ($\lfloor \cdot \rfloor$ is the floor function), we have that

$$\mathbf{D}_t^\eta f = \frac{1}{\Gamma(k-\eta)} \left(\frac{\partial}{\partial t} \right)^k \int_0^t \frac{1}{(t-s)^{\eta+1-k}} f(s) ds.$$

Let $\{G_\alpha(t)\}_{t \geq 0}$ denote the semigroup of operators defined via Fourier transform by

$$\widehat{G_\alpha(t) f} = E_\alpha(-t^\alpha |\xi|^2) \widehat{f}(\xi), \tag{1.4}$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is the Mittag-Leffler function. Taking $\alpha = 1$ and $\alpha = 2$, the operator $G_\alpha(t)$ has symbol $E_1(-t|\xi|^2) = e^{-t|\xi|^2}$ and $E_2(-t^2|\xi|^2) = \cos(|\xi|\, t)$, respectively, and (1.2) becomes the semilinear heat and wave equations. When $1 < \alpha < 2$ the semigroup possesses mixed properties of the heat and wave semigroups (see (2.15)-(2.17)). In original variables, the symbol $E_\alpha(-t^\alpha |\xi|^2)$ corresponds to the fundamental solution $\mathcal{K}_\alpha(x,t)$ of the linear part of (1.2) (see [10]), that is,

$$\mathcal{K}_\alpha(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} E_\alpha(-t^\alpha |\xi|^2) d\xi. \tag{1.5}$$

The problem (1.2)-(1.3) (or (1.1)) can be formally converted to the integral equation (see [17])

$$u(x,t) = G_\alpha(t) u_0(x) + B_\alpha(u), \tag{1.6}$$

with

$$B_\alpha(u)(t) = \int_0^t G_\alpha(t-s) \int_0^s R_{\alpha-1}(s-\tau) |u(\tau)|^\rho u(\tau) d\tau ds, \tag{1.7}$$

where $R_\eta(s) = s^{\eta-1}/\Gamma(\eta)$. Throughout this paper a mild solution for (1.2)-(1.3) (or (1.1)) is a function u satisfying (1.6).

Fractional differential equations have attracted the attention of many authors. For instance, with different definitions for ∂_t^α (or with the same one), the linear version of (1.2) $\partial_t^\alpha u = \Delta_x u + f(x,t)$ (or other related linear FPDEs) have been studied in [3],[6],[7],[8],[13],[10],[11],[12],[15],[16],[21],[35], where the reader can found results about existence, uniqueness and asymptotic behavior of solutions in L^p-spaces or in spaces of regular (continuous) functions, and results about fundamental solutions and their properties when $f \equiv 0$. In many of these references, the studied equation is presented in a time-integral form (like (1.1)) which corresponds a time fractional derivative notion (see e.g. [10],[11],[13],[35]). On the other hand, there are only a few results for the semilinear equation (1.2) and there is a lack of results in comparison with semilinear heat and wave equations. In this direction we mention the works [17],[27] in which the authors obtained results on global existence for (1.2) with, respectively, small initial data in the critical Lebesgue space $L^{\frac{n\rho}{2}}$ and in a subset of the critical Besov space $\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}$ (with $p > \frac{n\rho}{2}$). The local-in-time existence was also addressed in [17] regardless of the data size. Miao and

Yang [27] used certain L^p-space-time estimates due to [17] and their results also provide existence of self-similar solutions.

Models with fractional derivatives can naturally connect structurally different groups of PDEs and their mathematical analysis may give information about the transition (or loss) of basic properties from one to another. Two groups are the parabolic and hyperbolic PDEs whose well-posedness and asymptotic behavior theory presents a lot of differences. For instance, in L^p, weak-L^p, Besov-spaces, Morrey spaces and other ones, there is an extensive bibliography for global well-posedness and asymptotic behavior for nonlinear heat equations (and other parabolic equations), see e.g. [9], [19], [20], [24], [36] and references therein. On the other hand, for nonlinear wave equations, although there exist results in L^p [31, 34], weak$-L^p$ [26, 18] and Besov spaces [32], there is no results in Morrey spaces. The main reason is the loss of decay of the semigroup (and its time-derivative) associated to the free wave equation, namely $(\frac{\sin(|\xi|t)}{|\xi|t})^{\vee}$ and $(\cos(|\xi|\, t))^{\vee}$. So, it is natural to wonder what would be the behavior of the semilinear FPDE (1.2) in the framework of Morrey spaces, which presents a mixed parabolic-hyperbolic structure.

We show that the equation (1.6) is globally well-posedness for small initial data in the critical Morrey space $\mathcal{M}_{p,\mu}$ with $\mu = n - 2p/\rho$. Such spaces contain strongly singular functions and, for instance, they are lager than L^p and weak-L^p spaces in the critical case $p = \frac{n\rho}{2}$ (see (2.8)). There is no inclusion relation between Besov and Morrey space with the same scaling. In Remark 3.2, we comment how to derive a local-in-time version of our well-posedness result and discuss an alternative blow up scenario. We also investigate qualitative properties of solutions such as self-similarity, antisymmetry (and symmetry) and positivity (see Theorem 3.3). Moreover, we analyze the asymptotic stability of the solutions and thereby a class of asymptotically self-similar solutions is obtained.

Let us comment some interesting technical points. Due to the semigroup property (2.17), further restrictions appear in Theorem 3.1 in comparison with semilinear heat equations. Making the derivative index α go from $\alpha = 1$ to $\alpha = 2$, the estimates and corresponding restrictions become worse, and they are completely lost when α reaches the endpoint 2 (see Lemmas 4.2, 4.3 and 4.5). The proof of the pointwise estimate (4.2) shows that the "worst parcel" in (2.17) is the term $l_\alpha(\xi)$ (see (2.16)). In particular, notice that for $\alpha = 1$ (heat semigroup) the upper bound on parameter δ is not necessary, that is, one can take $\delta \in [0, \infty)$. Finally, based on above observations, our results and estimates suggest the following: "the semilinear wave equation ($\alpha = 2$) in $\mathbb{R}^n$ is not well-posed in Morrey spaces". The mathematical verification of this assertion seems to be an interesting open problem.

The manuscript is organized as follows. Some basic properties about Morrey spaces and Mittag-Leffler functions are reviewed in section 2. Our results are stated in section 3 and their proofs are performed in section 4.

2 Preliminary

In this section some properties about Morrey spaces and Mittag-Leffler functions are recalled.

2.1 Morrey spaces

The Morrey spaces and some basic properties of them are reviewed in the present subsection. For further details on theses spaces, the reader is referred to [20, 30, 36, 38].

Let $D_r(x_0)$ denote the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$. For two parameters $1 \leq p < \infty$ and $0 \leq \mu < n$, we define the Morrey spaces $\mathcal{M}_{p,\mu} = \mathcal{M}_{p,\mu}(\mathbb{R}^n)$ as the set of functions $f \in L^p(D_r(x_0))$ such that

$$\|f\|_{L^p(D_r(x_0))} \leq C\, r^{\frac{\mu}{p}}, \text{ for all } x_0 \in \mathbb{R}^n, \tag{2.1}$$

where $C > 0$ denotes a constant independent of x_0, r and f. The space $\mathcal{M}_{p,\mu}$ endowed with the norm

$$\|f\|_{p,\mu} = \sup_{r>0, x_0\in\mathbb{R}^n} r^{-\frac{\mu}{p}} \|f\|_{L^p(D_r(x_0))} \tag{2.2}$$

is a Banach space. For $s \in \mathbb{R}$ and $1 \leq p < \infty$, the homogeneous Sobolev-Morrey space $\mathcal{M}^s_{p,\mu} = (-\Delta)^{-s/2}\mathcal{M}_{p,\mu}$ is a Banach space with norm

$$\|f\|_{\mathcal{M}^s_{p,\mu}} = \left\|(-\Delta)^{s/2} f\right\|_{p,\mu}. \tag{2.3}$$

Taking $p = 1$, $\|f\|_{L^1(D_r(x_0))}$ stands for the total variation of f on $D_r(x_0)$ and $\mathcal{M}_{1,\mu}$ is a space of signed measures. In particular, when $\mu = 0$, $\mathcal{M}_{1,0} = \mathcal{M}$ is the space of finite measures. For $p > 1$, $\mathcal{M}_{p,0} = L^p$ and $\mathcal{M}^s_{p,0} = \dot{H}^s_p$ is the homogeneous Sobolev space. With the natural adaptation in (2.2) for $p = \infty$, the space L^∞ corresponds to $\mathcal{M}_{\infty,\mu}$.

Morrey spaces present the following scaling

$$\|f(\lambda x)\|_{p,\mu} = \lambda^{-\frac{n-\mu}{p}} \|f\|_{p,\mu} \tag{2.4}$$

and

$$\|f(\lambda x)\|_{\mathcal{M}^s_{p,\mu}} = \lambda^{s-\frac{n-\mu}{p}} \|f(x)\|_{\mathcal{M}^s_{p,\mu}}, \tag{2.5}$$

where the exponent $s - \frac{n-\mu}{p}$ is called scaling index. We have that

$$(-\Delta)^{l/2}\mathcal{M}^s_{p,\mu} = \mathcal{M}^{s-l}_{p,\mu}. \tag{2.6}$$

Let us define the closed subspace of $\mathcal{M}_{p,\mu}$ (denoted by $\ddot{\mathcal{M}}_{p,\mu}$) by means of the property $f \in \ddot{\mathcal{M}}_{p,\mu}$ if and only if

$$\|f(\cdot - y) - f(\cdot)\|_{p,\mu} \to 0 \text{ as } y \to 0. \tag{2.7}$$

This subspace is useful to deal with semigroup of convolution operators when the respective kernels present a suitable polynomial decay at infinity. In general, such semigroups are only weakly continuous at $t = 0^+$ in $\mathcal{M}_{p,\mu}$, but they are C_0-semigroups in $\ddot{\mathcal{M}}_{p,\mu}$, as it is the case of $\{G_\alpha(t)\}_{t\geq 0}$. This property is important in order to derive local-in-time well-posedness for PDEs (see Remark 3.2).

Morrey spaces contain Lebesgue and Marcinkiewicz spaces with the same scaling indexes. Precisely, we have the continuous proper inclusions

$$L^q(\mathbb{R}^n) \subset L^{q,\infty}(\mathbb{R}^n) \subset \mathcal{M}_{p,\mu}(\mathbb{R}^n) \tag{2.8}$$

where $p < q$ and $\mu = n(q-p)/q$ (see e.g. [28] or [1]).

In the next lemma, we remember some important inequalities and inclusions in Morrey spaces, see e.g. [20, 36].

Lemma 2.1. *Suppose that $s_1, s_2 \in \mathbb{R}$, $1 \leq p, q, r < \infty$ and $0 \leq \lambda, \mu, \upsilon < n$.*

(i) (Inclusion) If $p \leq q$ and $\frac{n-\mu}{p} = \frac{n-\lambda}{q}$ then

$$\mathcal{M}_{q,\lambda} \subset \mathcal{M}_{p,\mu} \tag{2.9}$$

(ii) (Sobolev type embedding) If $p \le q$*,* $s_2 \ge s_1$ *and* $s_2 - \frac{n-\mu}{p} = s_1 - \frac{n-\mu}{q}$ *then*

$$\mathcal{M}^{s_2}_{p,\mu} \subset \mathcal{M}^{s_1}_{q,\mu} \tag{2.10}$$

(iii) *(Hölder inequality) If* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and* $\frac{\upsilon}{r} = \frac{\lambda}{p} + \frac{\mu}{q}$ *then* $fg \in \mathcal{M}_{r,\upsilon}$ *and*

$$\|fg\|_{r,\upsilon} \leqslant \|f\|_{p,\lambda}\|g\|_{q,\mu}. \tag{2.11}$$

(iv) *(Homogeneous function) Let* $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$*,* $0 < d < n$ *and* $1 \le r < n/d$*. Then* $\Omega(x/|x|)|x|^{-d} \in \mathcal{M}_{r,n-dr}$*.*

We finish this subsection by recalling estimates for certain multiplier operators on $\mathcal{M}^{s}_{p,\mu}$ (see e.g. [23, 24, 36]).

Lemma 2.2. *Let* $m, s \in \mathbb{R}$*,* $1 < p < \infty$ *and* $0 \le \mu < n$ *and* $F(\xi) \in C^{[n/2]+1}(\mathbb{R}^n\backslash\{0\})$*. Assume that there is* $A > 0$ *such that*

$$\left|\frac{\partial^k F}{\partial \xi^k}(\xi)\right| \le A\,|\xi|^{m-|k|}, \tag{2.12}$$

for all $k \in (\mathbb{N} \cup \{0\})^n$ *with* $|k| \le [n/2] + 1$ *and for all* $\xi \neq 0$*. Then the multiplier operator* $F(D)$ *on* $\mathcal{S}'/\mathcal{P}$ *is bounded from* $\mathcal{M}^{s}_{p,\mu}$ *to* $\mathcal{M}^{s-m}_{p,\mu}$ *and the following estimate hold true*

$$\|F(D)f\|_{\mathcal{M}^{s-m}_{p,\mu}} \le C\,\|f\|_{\mathcal{M}^{s}_{p,\mu}}, \tag{2.13}$$

where $C > 0$ *is a constant independent of* f*, and the set* $\mathcal{S}'/\mathcal{P}$ *is the one of equivalence classes in* $\mathcal{S}'$ *modulo polynomials with* n *variables.*

2.2 Mittag-Leffler function

This part is devoted to review basic properties for Mittag-Leffler functions, the symbol $E_\alpha(-t^\alpha|\xi|^2)$ and the fundamental solution $\mathcal{K}_\alpha$ (see (1.5)). These can be found in [10] and [17], when $n = 1$ and $n \ge 2$, respectively.

The Mittag-Leffler function $E_\alpha(z)$ is defined via power series by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{2.14}$$

where $\Gamma(\alpha)$ is the Gamma function. In what follows, we recall some functions which is useful to handle the symbol of the semigroup $G_\alpha(t)$ (see (1.4)). For $1 < \alpha < 2$, let us set

$$a_\alpha(\xi) = |\xi|^{\frac{2}{\alpha}} e^{\frac{i\pi}{\alpha}}, \quad b_\alpha(\xi) = |\xi|^{\frac{2}{\alpha}} e^{-\frac{i\pi}{\alpha}}, \text{ for } \xi \in \mathbb{R}^n, \tag{2.15}$$

and

$$l_\alpha(\xi) = \begin{cases} \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{|\xi|^2 s^{\alpha-1} e^{-s}}{s^{2\alpha} + 2|\xi|^2 s^\alpha \cos(\alpha\pi) + |\xi|^4} ds & \text{if } \xi \neq 0 \\ 1 - \frac{2}{\alpha}, & \text{if } \xi = 0. \end{cases} \tag{2.16}$$

Lemma 2.3. *Let* $1 < \alpha < 2$ *and* $\mathcal{K}_\alpha$ *be as in (1.5). We have that*

$$E_\alpha(-|\xi|^2) = \frac{1}{\alpha}(\exp(a_\alpha(\xi)) + \exp(b_\alpha(\xi))) + l_\alpha(\xi) \tag{2.17}$$

and

$$\frac{\partial^k \mathcal{K}_\alpha}{\partial x_i^k}(x,t) = \lambda^{n+k}\frac{\partial^k}{\partial x_i^k}\mathcal{K}_\alpha(\lambda x, \lambda^{\frac{2}{\alpha}} t), \tag{2.18}$$

for all $\lambda > 0$. *Moreover,* $\mathcal{K}_\alpha(x,t) \geq 0$, $P_\alpha(|x|,1) = \alpha\mathcal{K}_\alpha(x,1)$ *is a probability measure, and*

$$\|\mathcal{K}_\alpha(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \frac{1}{\alpha},\ \textit{for all } t > 0. \tag{2.19}$$

Proof. Except for (2.18), all properties contained on the statement can be found in [10] and [17] when $n = 1$ and $n \geq 2$, respectively.

In order to prove (2.18), we use Fourier inversion and (1.5) to obtain

$$\begin{aligned}\frac{\partial^k \mathcal{K}_\alpha}{\partial x_i^k}(x,t) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}(-i\xi_i)^k E_\alpha(-t^\alpha|\xi|^2)d\xi = t^{-n\frac{\alpha}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\frac{y}{\sqrt{t^\alpha}}}(-it^{-\frac{\alpha}{2}}y_i)^k E_\alpha(-|y|^2)dy \\ &= t^{-\frac{\alpha}{2}(n+k)}\int_{\mathbb{R}^n} e^{i\frac{x}{\sqrt{t^\alpha}}\cdot y}(-iy_i)^k E_\alpha(-|y|^2)dy = t^{-\frac{\alpha}{2}(n+k)}\frac{\partial^k \mathcal{K}_\alpha}{\partial x_i^k}(\frac{x}{\sqrt{t^\alpha}},1). \end{aligned} \tag{2.20}$$

The desired identity follows by taking $\lambda = 1/\sqrt{t^\alpha}$ in (2.20).

■

3 Functional setting and results

We shall employ the Kato-Fujita method (see [19]) to integral equation (1.6), which should be understood in the Bochner sense in Morrey spaces.

In what follows, we perform a scaling analysis in order to choose the correct indexes for Kato-Fujita spaces. A simple computation by using (1.2) shows that the indexes $k_1 = 2/\rho$ and $k_2 = 2/\alpha$ are the unique ones such that the function u_λ given by

$$u_\lambda(x,t) = \lambda^{k_1}u(\lambda x, \lambda^{k_2}t) \tag{3.1}$$

is a solution of (1.2), for each $\lambda > 0$, whenever u is also. The scaling map for (1.2) is defined by

$$u(x,t) \to u_\lambda(x,t). \tag{3.2}$$

Making $t \to 0^+$ in (3.2) one obtains the following scaling for the initial condition

$$u_0(x) \to \lambda^{2/\rho}u_\lambda(x). \tag{3.3}$$

Solutions invariant by (3.2), that is

$$u(x,t) = u_\lambda(x,t) \text{ for all } \lambda > 0, \tag{3.4}$$

are called self-similar ones. Since we are interested in such solutions, it is suitable to consider critical spaces for $u(x,t)$ and u_0, i.e., the ones whose norms are invariant by (3.2) and (3.3), respectively.

Consider the parameters

$$\mu = n - 2p/\rho \text{ and } \beta = \frac{\alpha}{\rho} - \alpha\frac{n-\mu}{2q}, \tag{3.5}$$

and let $BC((0,\infty),X)$ stands for the class of bounded and continuous functions from $(0,\infty)$ to a Banach space X. We take u_0 belonging to the critical space $\mathcal{M}_{p,\mu}$ and study (1.6) in the Kato-Fujita type space

$$H_q = \{u(x,t) \in BC((0,\infty);\mathcal{M}_{p,\mu}) \,:\, t^{\beta}u \in BC((0,\infty);\mathcal{M}_{q,\mu})\}, \tag{3.6}$$

which is a Banach space with the norm

$$\|u\|_{H_q} = \sup_{t>0} \|u(\cdot,t)\|_{p,\mu} + \sup_{t>0} t^{\beta}\|u(\cdot,t)\|_{q,\mu}. \tag{3.7}$$

Notice that the norm (3.7) is invariant by scaling transformation (3.2).

From Lemma 2.1, a typical data belonging to $\mathcal{M}_{p,\mu}$ is the homogeneous function

$$u_0(x) = \Omega(\frac{x}{|x|})\,|x|^{-\frac{2}{\rho}} \tag{3.8}$$

where $1 \leq p < \frac{n\rho}{2}$ and Ω is a bounded function on sphere $\mathbb{S}^{n-1}$. We refer the book [14, chapter 3] for more details about self-similar solutions and PDE's.

Our well-posedness result reads as follows.

Theorem 3.1 (Well-posedness). *Let $0 < \rho < \infty$, $1 < \alpha < 2$, $1 < p < \frac{n\rho}{2}$, and $\mu = n - 2p/\rho$. Suppose that $\frac{n-\mu}{p} - \frac{n-\mu}{q} < 2$ and*

$$1 - \frac{1}{\alpha}\frac{\rho}{\rho+1} < \frac{p}{q} < \frac{1}{\alpha} \quad \text{and} \quad \frac{\rho p}{p-1} < q < \infty. \tag{3.9}$$

(i) (Existence and uniqueness) There exist $\varepsilon > 0$ and $\delta = \delta(\varepsilon)$ such that if $\|u_0\|_{p,\mu} \leq \delta$ then the equation (1.1) has a mild solution $u \in H_q$, which is the unique one in the ball $D_{2\varepsilon} = \{u \in H_q; \|u\|_{H_q} \leq 2\varepsilon\}$. Moreover, $u \rightharpoonup u_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0^+$

(ii) (Continuous dependence on data) Let $\mathcal{I}_0 = \{u_0 \in \mathcal{M}_{p,\mu}; \|u\|_{p,\mu} \leq \delta\}$. The data-solution map is Lipschitz continuous from $\mathcal{I}_0$ to $D_{2\varepsilon}$.

Remark 3.2.

(i) With a slight adaptation of the proof of Theorem 3.1, we could treat more general nonlinearities. Precisely, one could consider (1.1) and (1.2) with $f(u)$ instead of $u\,|u|^{\rho}$, where $f \in C(\mathbb{R})$, $f(0) = 0$ and there is $C > 0$ such that

$$|f(a) - f(b)| \leq C|a-b|(|a|^{\rho} + |b|^{\rho}), \text{ for all } a,b \in \mathbb{R}.$$

(ii) (Local-in-time well-posedness) A local version of Theorem 3.1 holds true by replacing the smallness condition on initial data by a smallness one on existence time $T > 0$. Here we should consider the local-in-time space

$$H_{q,T} = \{u(x,t) \in BC((0,T);\mathcal{M}_{p,\mu}) \,:\, \lim\sup_{t\to 0^+} t^{\beta}\|u(\cdot,t)\|_{q,\mu} = 0\}, \tag{3.10}$$

and $u_0 \in \mathcal{M}_{p,\mu}$ such that $\lim\sup_{t\to 0^+} t^{\beta}\|G_\alpha(t)u_0\|_{q,\mu} = 0$. In particular, this condition is verified when u_0 belongs to $\ddot{\mathcal{M}}_{p,\mu}$ (see (2.7)). Thus we need to restrict ourselves to the suitable subspace $\ddot{\mathcal{M}}_{p,\mu} \subsetneq \mathcal{M}_{p,\mu}$ which is the maximal closed one where the group of translation is continuous (see [20, Lemma 3.1]). As already pointed in subsection 2.1, the main reason for that is the semigroup $\{G_\alpha(t)\}_{t\geq 0}$ is not strongly continuous at $t = 0^+$ on $\mathcal{M}_{p,\mu}$.

(iii) (Alternative blow up) The subspace $\ddot{\mathcal{M}}_{p,\mu}$ does not contain, in particular, homogeneous functions and so we are not able to obtain local solutions for arbitrary data $u_0 \in \mathcal{M}_{p,\mu}$ as in Theorem 3.1 by means of the approach employed in the present paper. These data correspond to self-similar solutions (see Theorem 3.3 below). However, in view of item (ii) of this remark, if $u_0 \in \ddot{\mathcal{M}}_{p,\mu}$ then the local-in-time version of Theorem 3.1 gives a solution $u \in C([0,T_{\max});\mathcal{M}_{p,\mu})$, where $T_{\max} > 0$ is the maximal existence time. Moreover, an alternative blow up holds true, that is, there holds either $T_{\max} = \infty$ or else $\lim_{t\to T_{\max}^-} \|u(\cdot,t)\|_{\mathcal{M}_{p,\mu}} = \infty$ with $T_{\max} < \infty$. We refer the reader to [2, 4, 25] for more results about blow up (self-similar or not) for nonlinear diffusion equations.

Let $O(n)$ be the orthogonal matrix group in $\mathbb{R}^n$ and let $\mathcal{G}$ be a subset of $O(n)$. A function h is said symmetric and antisymmetric under the action of $\mathcal{G}$ when $h(x) = h(Mx)$ and $h(x) = -h(Mx)$, respectively, for every $M \in \mathcal{G}$.

Theorem 3.3. *Under the hypotheses of Theorem 3.1.*

(i) *(Self-similarity) If u_0 is a homogeneous function of degree $-\frac{2}{\rho}$, then the mild solution given in Theorem 3.1 is self-similar.*

(ii) *(Symmetry and antisymmetry) The solution $u(x,t)$ is antisymmetric (resp. symmetric) for $t > 0$, when u_0 is antisymmetric (resp. symmetric) under $\mathcal{G}$.*

(iii) *(Positivity) If $u_0 \not\equiv 0$ and $u_0(x) \geq 0$ (resp. $u_0(x) \leq 0$) then u is positive (resp. negative).*

Remark 3.4. *(Special examples of symmetry and antisymmetry)*

(i) *The case $\mathcal{G} = O(n)$ corresponds to radial symmetry. Therefore, it follows from Theorem 3.3 (ii) that if u_0 is radially symmetric then $u(x,t)$ is radially symmetric for $t > 0$.*

(ii) *Let $Mx = -x$ be the reflection over the origin and let $I_{\mathbb{R}^n}$ be the identity map. The case $\mathcal{G} = \{I_{\mathbb{R}^n}, M\}$ corresponds to parity of functions, that is, $h(x)$ is even and odd when $h(x) = h(-x)$ and $h(x) = -h(-x)$, respectively. So, from Theorem 3.3 (ii), we have that the solution $u(x,t)$ is even (resp. odd) for $t > 0$, when $u_0(x)$ is even (resp. odd).*

The next theorem gives a criterion for the asymptotic stability of solutions and provides a class of asymptotically self-similar solutions.

Theorem 3.5. *Under the hypotheses of Theorem 3.1. Let u and v be two global mild solutions for (1.1) given by Theorem 3.1, with respective data u_0 and v_0. We have that*

$$\lim_{t\to+\infty} \|u(\cdot,t) - v(\cdot,t)\|_{p,\mu} = \lim_{t\to+\infty} t^{\beta}\|u(\cdot,t) - v(\cdot,t)\|_{q,\mu} = 0 \tag{3.11}$$

if and only if

$$\lim_{t\to+\infty} (\|G_\alpha(t)(u_0 - v_0)\|_{p,\mu} + t^{\beta}\|G_\alpha(t)(u_0 - v_0)\|_{q,\mu}) = 0. \tag{3.12}$$

Remark 3.6. *(Asymptotically self-similar solutions) In Theorem 3.5, let $v_0 = u_0 + \varphi$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ small enough and suppose that u_0 is a function homogeneous of degree $-\frac{2}{\rho}$. We have that $\varphi = v_0 - u_0$ satisfies (3.12) and then the solution $v(x,t)$ converges in the sense of (3.11) to the self-similar solution u as $t \to +\infty$, i.e., it is asymptotically self-similar.*

4 Proofs of theorems

We start by recalling an elementary fixed point lemma whose proof can be found in [9].

Lemma 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and $0 < \rho < \infty$. Suppose that $B : X \to X$ satisfies $B(0) = 0$ and*

$$\|B(x) - B(z)\| \leq K\|x - z\|(\|x\|^{\rho} + \|z\|^{\rho}).$$

Let $R > 0$ be the unique positive root of $2^{\rho+1} K R^{\rho} - 1 = 0$. Given $0 < \varepsilon < R$ and $y \in X$ such that $\|y\| \leq \varepsilon$, there exists a solution $x \in X$ for the equation $x = y + B(x)$ which is the unique one in the closed ball $D_{2\varepsilon} = \{z \in X; \|z\| \leq 2\varepsilon\}$. Moreover, if $\|\bar{y}\| \leq \varepsilon$ and $\bar{x} \in D_{2\varepsilon}$ satisfies the equation $\bar{x} = \bar{y} + B(\bar{x})$ then

$$\|x - \bar{x}\| \leq \frac{1}{1 - 2^{\rho+1} K \varepsilon^{\rho}} \|y - \bar{y}\|. \tag{4.1}$$

4.1 Linear estimates

The aim of this subsection is to derive estimates for the semigroup $G_\alpha(t)$. For that matter we will need pointwise estimates for the fundamental solution $\mathcal{K}_\alpha$ in Fourier variables.

Lemma 4.2. *Let $1 \leq \alpha < 2$ and $0 \leq \delta < 2$. There is $C := C(\alpha, \delta, n) > 0$ such that*

$$\left| \frac{\partial^k}{\partial \xi^k} \left[|\xi|^{\delta} E_\alpha(-|\xi|^2) \right] \right| \leq C\, |\xi|^{-|k|}, \tag{4.2}$$

for all $k \in (\mathbb{N} \cup \{0\})^n$ with $|k| \leq [n/2] + 1$ and for all $\xi \neq 0$.

Proof. In view of (2.17), the symbol $E_\alpha(-|\xi|^2)$ is composed by two parcel, namely

$$I_\alpha(\xi) = \frac{1}{\alpha} \exp(|\xi|^{\frac{2}{\alpha}} e^{\frac{i\pi}{\alpha}}) + \frac{1}{\alpha} \exp(|\xi|^{\frac{2}{\alpha}} e^{-\frac{i\pi}{\alpha}}) \tag{4.3}$$

and

$$l_\alpha(\xi) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{|\xi|^2 s^{\alpha-1} e^{-s}}{s^{2\alpha} + 2|\xi|^2 s^\alpha \cos(\alpha\pi) + |\xi|^4} ds \tag{4.4}$$

$$= \frac{\sin(\alpha\pi)}{\alpha\pi} \int_0^\infty \frac{\exp(-|\xi|^{\frac{2}{\alpha}} s^{\frac{1}{\alpha}})}{s^2 + 2s\cos(\alpha\pi) + 1} ds, \tag{4.5}$$

where the change $s \mapsto |\xi|^{\frac{2}{\alpha}} s^{\frac{1}{\alpha}}$ was used from (4.4) to (4.5). For $\delta > 0$, we have that

$$\begin{aligned} \left| \frac{\partial^k}{\partial \xi^k} \left[|\xi|^{\delta} I_\alpha(\xi) \right] \right| &\leq C(|\xi|^{\delta - |k|} + |\xi|^{\delta + \frac{2|k|}{\alpha}}) \exp(|\xi|^{\frac{2}{\alpha}} \cos(\frac{\pi}{\alpha})) \\ &\leq C\, |\xi|^{-|k|}. \end{aligned} \tag{4.6}$$

On the other hand,

$$\left| \frac{\partial^k}{\partial \xi^k} \left[|\xi|^{\delta} l_\alpha(\xi) \right] \right| \leq C \int_0^\infty \frac{g(\xi, s)}{s^2 + 2s\cos(\alpha\pi) + 1} ds,$$

where

$$g(\xi,s)=\left(|\xi|^{\delta-|k|}+|\xi|^{\delta-(|k|-1)}|\xi|^{(\frac{2-\alpha}{\alpha})}s^{\frac{1}{\alpha}}+|\xi|^{\delta-(|k|-2)}|\xi|^{(\frac{2-\alpha}{\alpha})2}s^{\frac{2}{\alpha}}+...+|\xi|^{\delta}|\xi|^{(\frac{2-\alpha}{\alpha})|k|}s^{\frac{|k|}{\alpha}}\right)\exp(-|\xi s^{\frac{1}{2}}|^{\frac{2}{\alpha}})$$

$$=|\xi|^{-|k|}\left(|\xi|^{\delta}+|\xi|^{\delta}|\xi|^{\frac{2}{\alpha}}s^{\frac{1}{\alpha}}+|\xi|^{\delta}|\xi|^{(\frac{2}{\alpha})2}s^{\frac{2}{\alpha}}+...+|\xi|^{\delta}|\xi|^{(\frac{2}{\alpha})|k|}s^{\frac{|k|}{\alpha}}\right)\exp(-|\xi s^{\frac{1}{2}}|^{\frac{2}{\alpha}})$$

$$=s^{-\frac{\delta}{2}}|\xi|^{-|k|}\left(\left|\xi s^{\frac{1}{2}}\right|^{\delta}+\left|\xi s^{\frac{1}{2}}\right|^{\frac{2}{\alpha}+\delta}+\left|\xi s^{\frac{1}{2}}\right|^{\frac{4}{\alpha}+\delta}+...+\left|\xi s^{\frac{1}{2}}\right|^{\frac{2|k|}{\alpha}+\delta}\right)\exp(-|\xi s^{\frac{1}{2}}|^{\frac{2}{\alpha}})$$

$$\leq Cs^{-\frac{\delta}{2}}|\xi|^{-|k|}.$$

Therefore,

$$\left|\frac{\partial^k}{\partial\xi^k}\left[|\xi|^{\delta}l_\alpha(\xi)\right]\right|\leq C|\xi|^{-|k|}\int_0^\infty\frac{s^{-\frac{\delta}{2}}}{s^2+2s\cos(\alpha\pi)+1}ds,$$
$$\leq C|\xi|^{-|k|},\tag{4.7}$$

because $\delta<2$. Now the estimate (4.2) follows from (4.6) and (4.7).

■

In the sequel we prove key estimates on Morrey spaces for the semigroup $\{G_\alpha(t)\}_{t\geq 0}$.

Lemma 4.3. *Let $1\leq\alpha<2$, $1<p\leq q<\infty$, $0\leq\mu<n$, and $\frac{n-\mu}{p}-\frac{n-\mu}{q}<2$. There exists $C>0$ such that*

$$\|G_\alpha(t)f\|_{q,\mu}\leq Ct^{-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})}\|f\|_{p,\mu},\tag{4.8}$$

for all $f\in\mathcal{M}_{p,\mu}$.

Proof. Let $\delta=\frac{n-\mu}{p}-\frac{n-\mu}{q}$, $f_\lambda(x)=f(\lambda x)$ and $h_\alpha(x,t)$ defined through $\widehat{h_\alpha}(\xi,t)=|\xi|^{\delta}E_\alpha(-|\xi|^2t^\alpha)$. Consider the multiplier operators

$$F(D)f=[(-\Delta)^{\frac{\delta}{2}}G_\alpha(1)]f=h_\alpha(\cdot,1)*f(\cdot),$$
$$\left([(-\Delta)^{\frac{\delta}{2}}G_\alpha(t)]f\right)(x)=t^{-\delta\frac{\alpha}{2}}\left(h_\alpha(\cdot,1)*f_{t^{\alpha/2}}(\cdot)\right)_{t^{-\alpha/2}}(x)$$
$$=t^{-\delta\frac{\alpha}{2}}\left(F(D)(f_{t^{\alpha/2}})\right)_{t^{-\alpha/2}}(x),\tag{4.9}$$

where the symbol of $F(D)$ is $|\xi|^{\delta}E_\alpha(-|\xi|^2)$. Lemma 4.2 implies that $F(\xi)$ satisfies (2.12) with $m=0$. Then, we use (2.4) to obtain

$$\left\|\left(F(D)(f_{t^{\alpha/2}})\right)_{t^{-\alpha/2}}\right\|_{p,\mu}=t^{-\frac{\alpha}{2}(\frac{n-\mu}{p})}\left\|F(D)(f_{t^{\alpha/2}})\right\|_{p,\mu}$$
$$\leq Ct^{-\frac{\alpha}{2}(\frac{n-\mu}{p})}\left\|f_{t^{\alpha/2}}\right\|_{p,\mu}$$
$$=Ct^{-\frac{\alpha}{2}(\frac{n-\mu}{p})}t^{\frac{\alpha}{2}(\frac{n-\mu}{p})}\|f\|_{p,\mu}$$
$$=C\|f\|_{p,\mu}.\tag{4.10}$$

Now, using Sobelev embedding (2.10), and afterwards (4.9), we obtain

$$\|G_\alpha(t)f\|_{q,\mu}\leq\|G_\alpha(t)f\|_{\mathcal{M}^{\delta}_{p,\mu}}$$
$$=\left\|(-\Delta)^{\frac{\delta}{2}}G_\alpha(t)f\right\|_{p,\mu}$$
$$=t^{-\delta\frac{\alpha}{2}}\left\|\left(F(D)(f_{t^{\alpha/2}})\right)_{t^{-\alpha/2}}\right\|_{p,\mu}$$
$$\leq Ct^{-\frac{\alpha}{2}(\frac{n-\mu}{p}-\frac{n-\mu}{q})}\|f\|_{p,\mu},$$

because of (4.10).

∎

Using Lemma 4.3 we can estimate the H_q-norm in of the linear part in (1.6) by using the initial data $u_0 \in \mathcal{M}_{p,\mu}$.

Lemma 4.4. *Let $1 \leq \alpha < 2$, $0 < \rho < \infty$, $1 < p \leq \frac{n\rho}{2}$, $p \leq q < \infty$, $\mu = n - 2p/\rho$ and $\frac{n-\mu}{p} - \frac{n-\mu}{q} < 2$. There exists $L > 0$ such that*

$$\|G_\alpha(t)u_0\|_{H_q} \leq L\|u_0\|_{p,\mu}, \tag{4.11}$$

for all $u_0 \in \mathcal{M}_{p,\mu}$.

Proof. It follows from (4.8) that

$$\begin{aligned}\sup_{t>0}\|G_\alpha(t)u_0\|_{p,\mu} + \sup_{t>0} t^\beta\|G_\alpha(t)u_0\|_{q,\mu} &\leq C\left(\|u_0\|_{p,\mu} + \sup_{t>0} t^{\beta - \frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{q})}\|u_0\|_{p,\mu}\right)\\ &= C\|u_0\|_{p,\mu},\end{aligned}$$

and then we get (4.11).

∎

4.2 Nonlinear estimates

Firstly, let us recall the nonlinear parcel in (1.6) given by

$$B_\alpha(u)(t) = \int_0^t G_\alpha(t-s)\int_0^s R_{\alpha-1}(s-\tau)|u(\tau)|^\rho u(\tau)d\tau ds. \tag{4.12}$$

Before estimating B_α we recall a real number inequality. Given $\rho > 0$, there is $C > 0$ such that

$$|a\,|a|^\rho - b\,|b|^\rho| \leq C|a-b|(|a|^\rho + |b|^\rho), \text{ for all } a, b \in \mathbb{R}. \tag{4.13}$$

Lemma 4.5 (Nonlinear estimate)**.** *Under the assumptions of Theorem 3.1. There is a positive constant K such that*

$$\|B_\alpha(u) - B_\alpha(v)\|_{H_q} \leq K\|u-v\|_{H_q}(\|u\|_{H_q}^\rho + \|v\|_{H_q}^\rho), \tag{4.14}$$

for all $u, v \in H_q$.

Proof. The proof is performed in two steps.

First step. We start by estimating the norm $\|\cdot\|_{q,\mu}$. Thanks to Lemma 4.3, Hölder inequality (2.11) and the inequality (4.13), we obtain

$$\begin{aligned}\|B(u)(t) - B(v)(t)\|_{q,\mu} &\leq \int_0^t \|G_\alpha(t-s)\int_0^s R_{\alpha-1}(s-\tau)[f(u(\tau)) - f(v(\tau))]d\tau\|_{q,\mu}ds\\ &\leq C\int_0^t (t-s)^{\gamma_1}\|\int_0^s R_{\alpha-1}(s-\tau)[f(u(\tau)) - f(v(\tau))]d\tau\|_{\frac{q}{\rho+1},\mu}ds\\ &\leq C\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\|f(u(\tau)) - f(v(\tau))\|_{\frac{q}{\rho+1},\mu}d\tau ds\\ &\leq C\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\|u-v\|_{q,\mu}(\|u\|_{q,\mu}^\rho + \|v\|_{q,\mu}^\rho)d\tau ds,\end{aligned} \tag{4.15}$$

where $R_\alpha(s) = s^{\alpha-1}/\Gamma(\alpha)$, $f(u(\tau)) = |u(\tau)|^\rho u(\tau)$ and $\gamma_1 = \frac{\alpha}{2}(\frac{n-\mu}{q} - \frac{n-\mu}{q/(\rho+1)}) = -\alpha\rho\frac{n-\mu}{2q}$. Now we define $\theta(s)$ by

$$\begin{aligned}\theta(s) &= \int_0^s R_{\alpha-1}(s-\tau)\|u(\tau)-v(\tau)\|_{q,\mu}(\|u(\tau)\|_{q,\mu}^\rho + \|v(\tau)\|_{q,\mu}^\rho)d\tau \\ &= \int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta(\rho+1)}\tau^\beta\|u(\tau)-v(\tau)\|_{q,\mu}(\tau^{\beta\rho}\|u(\tau)\|_{q,\mu}^\rho + \tau^{\beta\rho}\|v(\tau)\|_{q,\mu}^\rho)d\tau,\end{aligned} \tag{4.16}$$

and rewrite (4.15) as

$$\|B(u)(\cdot,t) - B(v)(\cdot,t)\|_{q,\mu} \le C\int_0^t (t-s)^{\gamma_1}\theta(s)ds. \tag{4.17}$$

The function $\theta(s)$ can be estimated as

$$\theta(s) \le \int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta(\rho+1)}d\tau\,\|u-v\|_{H_q}(\|u\|_{H_q}^\rho + \|v\|_{H_q}^\rho). \tag{4.18}$$

Making the change of variables $\tau = zs$ and afterwards $s = t\omega$, we get

$$\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta(\rho+1)}d\tau ds = C\int_0^t (t-s)^{\gamma_1}s^\vartheta\int_0^1(1-z)^{\alpha-2}z^{-\beta(\rho+1)}dzds \tag{4.19}$$

$$= Ct^{\vartheta+\gamma_1+1}\int_0^1(1-\omega)^{\gamma_1}\omega^\vartheta d\omega, \tag{4.20}$$

where $\vartheta = \alpha - 1 - \beta(\rho+1)$ and

$$\vartheta + \gamma_1 + 1 = \alpha - 1 - \beta(\rho+1) - \alpha\rho\frac{n-\mu}{2q} + 1 = \beta\rho - \beta(\rho+1) = -\beta.$$

The convergence of the beta functions appearing in (4.19) and (4.20) follows at once from the hypotheses on parameters. Therefore

$$\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta(\rho+1)}d\tau ds = Ct^{-\beta}. \tag{4.21}$$

It follows from (4.17), (4.18) and (4.21) that

$$\sup_{t>0} t^\beta\|B(u)(t) - B(v)(t)\|_{q,\mu} \le K_1\|u-v\|_{H_q}(\|u\|_{H_q}^\rho + \|v\|_{H_q}^\rho). \tag{4.22}$$

Second step. Here we deal with the norm $\|\cdot\|_{p,\mu}$. Notice that we can choose $r > 1$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q/\rho}$. Then

$$\begin{aligned}&\|B(u)(t) - B(v)(t)\|_{p,\mu} \\ &\le C\int_0^t (t-s)^{\gamma_2}\int_0^s R_{\alpha-1}(s-\tau)\,\||u-v|\,(|u|^\rho + |v|^\rho)\|_{r,\mu}\,d\tau ds \\ &\le C\int_0^t (t-s)^{\gamma_2}\int_0^s R_{\alpha-1}(s-\tau)\|u-v\|_{p,\mu}(\|u\|_{q,\mu}^\rho + \|v\|_{q,\mu}^\rho)d\tau ds \qquad (4.23) \\ &= \int_0^t (t-s)^{\gamma_2}\int_0^s R_{\alpha-1}(s-\tau)\tau^{-\rho\beta}\,\|u-v\|_{p,\mu}(\tau^{\rho\beta}\|u\|_{q,\mu}^\rho + \tau^{\rho\beta}\|v\|_{q,\mu}^\rho)d\tau ds \\ &\le C\int_0^t (t-s)^{\gamma_2}\int_0^s (s-\tau)^{\alpha-2}\tau^{-\rho\beta}d\tau ds\,\|u-v\|_{H_q}(\|u\|_{H_q}^\rho + \|v\|_{H_q}^\rho) \qquad (4.24) \\ &= K_2\,\|u-v\|_{H_q}(\|u\|_{H_q}^\rho + \|v\|_{H_q}^\rho), \qquad (4.25)\end{aligned}$$

where $\gamma_2 = \frac{\alpha}{2}(\frac{n-\mu}{p} - \frac{n-\mu}{r}) = \rho\beta - \alpha = \gamma_1$, and we have proceeded similarly to (4.19) from (4.24) to (4.25).

In view of (3.7), the inequalities (4.22) and (4.25) together imply the desired estimate (4.14) with $K = K_1 + K_2$.

$\blacksquare$

4.3 Proof of Theorem 3.1

Part (i): Let $R = \left(\frac{1}{2^{\rho+1}K}\right)^{\frac{1}{\rho}}$, $0 < \varepsilon < R$ and $\delta = \frac{\varepsilon}{L}$, where $L > 0$ and $K > 0$ are the constants obtained in Lemmas 4.4 and 4.5, respectively. Taking $y = G_\alpha(t)u_0$, Lemma 4.4 yields

$$\|y\|_{H_q} = \|G_\alpha(t)u_0\|_{H_q} \leq L \|u_0\|_{p,\mu} \leq \varepsilon,$$

because $\|u_0\|_{p,\mu} \leq \delta$. Thanks to the inequality (4.14), we can take $X = H_q$ in Lemma 4.1 and conclude that there exists a global mild solution $u \in H_q$ for (1.1), which is the unique solution satisfying $\|u\|_{H_q} \leq 2\varepsilon$.

The convergence $u(t) \rightharpoonup u_0$ in $\mathcal{D}'(\mathbb{R}^n)$ follows similarly from standard arguments found, for instance, in [37].

Part (ii): Let u, v be two mild solutions with initial data u_0, v_0 belonging to $\mathcal{I}_0$, respectively. We can use (4.1) and Lemma 4.4 to estimate

$$\begin{aligned}
\|u - v\|_{H_q} &\leq \frac{1}{1 - 2^{\rho+1}K\varepsilon^\rho} \|G_\alpha(t)u_0 - G_\alpha(t)v_0\|_{p,\mu} \\
&= \frac{1}{1 - 2^{\rho+1}K\varepsilon^\rho} \|G_\alpha(t)(u_0 - v_0)\|_{p,\mu} \\
&\leq \frac{L}{1 - 2^{\rho+1}K\varepsilon^\rho} \|u_0 - v_0\|_{p,\mu},
\end{aligned}$$

as desired.

$\blacksquare$

4.4 Proof of Theorem 3.3

Part (i): Let $\Phi(x,t) = G_\alpha(t)u_0$. Using that u_0 is homogeneous of degree $-\frac{2}{\rho}$ and (2.18), it is not to difficult to check that

$$\Phi(x,t) = \Phi_\lambda(x,t) = \lambda^{\frac{2}{\rho}} \Phi(\lambda x, \lambda^{\frac{2}{\alpha}} t),$$

for every $\lambda > 0$. Also, we have that $[B_\alpha(u,u)]_\lambda = B_\alpha(u_\lambda, u_\lambda)$, for all $\lambda > 0$,where $[\Omega(x,t)]_\lambda$ stands for $\lambda^{\frac{2}{\rho}} \Omega(\lambda x, \lambda^{\frac{2}{\alpha}} t)$. It follows that u_λ is a solution for (1.6) because u is a solution. Due to the norm in H_q is scaling invariant, we have that $\|u_\lambda\|_{H_q} = \|u\|_{H_q} \leq 2\varepsilon$. From the uniqueness result contained in Theorem 3.1 (i), it follows that

$$u(x,t) \equiv u_\lambda(x,t), \text{ for every } \lambda > 0,$$

that is, u is self-similar.

Part (ii): Let $M \in \mathcal{G}$ and u_0 be antisymmetric. Using that M is orthogonal, the property $u_0(Mx) = -u_0(x)$ can be expressed in Fourier variables as

$$-\widehat{u_0}(\xi) = [u_0(Mx)]^\wedge(\xi) = \widehat{u_0}(M^{-1}\xi) \tag{4.26}$$

Again denoting $\Phi(x,t)=G_\alpha(t)u_0$, we obtain from (4.26) that

$$\begin{aligned}[\Phi(Mx,t)]^\wedge(\xi) &= E_\alpha(-t^\alpha|M^{-1}\xi|^2)\widehat{u_0}(M^{-1}\xi)\\ &= -E_\alpha(-t^\alpha|\xi|^2)\widehat{u_0}(\xi)\\ &= -\widehat{\Phi(x,t)}(\xi),\end{aligned}$$

which shows that $G_\alpha(t)u_0$ is antisymmetric for each fixed $t>0$. Similarly, we can show that $B_\alpha(u)$ is antisymmetric whether u is also. So, employing an induction argument, one can prove that each element u_k of the Picard sequence

$$u_1(x,t) = \Phi(x,t) \tag{4.27}$$
$$u_k(x,t) = \Phi(x,t) + B_\alpha(u_{k-1})(x,t),\ k=2,3,\cdots \tag{4.28}$$

is antisymmetric. Since $u_k \to u$ in H_q, then the sequence (4.27)-(4.28) also converges (up a subsequence) a.e. $x\in\mathbb{R}^n$ and $t>0$. It follows that $u(x,t)$ is antisymmetric for $t>0$, because pointwise convergence preserves antisymmetry. We finish this part by observing that the proof of the symmetric property is analogous.

Part (iii): We will only prove the statement concerning positive solutions, because the one about negative solutions follows similarly.

Let u_k be the sequence (4.27)-(4.28). From Lemma 2.3, we have that

$$u_1(x,t) = G_\alpha(t)u_0 = \mathcal{K}_\alpha(\cdot,t)*u_0,$$

with positive kernel $\mathcal{K}_\alpha(x,t)\geq 0$. Hence if $u_0\geq 0$ with $u_0\not\equiv 0$ a.e. in $\mathbb{R}^n$ then $u_1(x,t)>0$ in $\mathbb{R}^n$, for each $t>0$. Moreover, $B_\alpha(u)\geq 0$ whenever $u(x,t)\geq 0$. Therefore, an induction argument on k shows that $u_k(x,t)>0$, for all $k\in\mathbb{N}$. We have already seen in the last proof that (up a subsequence) $u_k\to u$ a.e. $x\in\mathbb{R}^n$ and $t>0$. Therefore the limit $u(x,t)\geq 0$ a.e. $x\in\mathbb{R}^n$ and $t>0$. But, since u verifies (1.6), we get

$$u = u_1(x,t) + B_\alpha(u) \geq u_1(x,t) > 0,$$

for a.e. $x\in\mathbb{R}^n$ and $t>0$.

■

4.5 Proof of Theorem 3.5

We only show that (3.12) implies (3.11), because the converse statement follows analogously. Subtracting the integral equations verified by u and v, and then taking the norms $t^\beta\|\cdot\|_{q,\mu}$ and $\|\cdot\|_{p,\mu}$ we obtain

$$\begin{aligned}t^\beta\|u(\cdot,t)-v(\cdot,t)\|_{q,\mu} &\leq t^\beta\|G_\alpha(t)(u_0-v_0)\|_{q,\mu} + t^\beta\|B_\alpha(u)-B_\alpha(v)\|_{q,\mu}\\ &:= t^\beta\|G_\alpha(t)(u_0-v_0)\|_{q,\mu} + I_1(t)\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}\|u(\cdot,t)-v(\cdot,t)\|_{p,\mu} &\leq \|G_\alpha(t)(u_0-v_0)\|_{p,\mu} + \|B_\alpha(u)-B_\alpha(v)\|_{p,\mu}\\ &\leq \|G_\alpha(t)(u_0-v_0)\|_{p,\mu} + I_2(t).\end{aligned} \tag{4.30}$$

Recalling that $\|u\|_{H_q}\leq 2\varepsilon$ and $\|v\|_{H_q}\leq 2\varepsilon$, and the inequality (4.15), we can estimate the term $I_1(t)$ as

$$\begin{aligned}I_1(t) &\leq Ct^\beta\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\|u(\tau)-v(\tau)\|_{q,\mu}\|\,|u(\tau)|^\rho+|v(\tau)|^\rho\|_{q,\mu}d\tau ds\\ &\leq t^\beta 2(2\varepsilon)^\rho C\int_0^t (t-s)^{\gamma_1}\int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta(\rho+1)}\Sigma_1(\tau)d\tau ds,\end{aligned} \tag{4.31}$$

where $\Sigma_1(\tau) = t^{\beta}\|u(\tau) - v(\tau)\|_{q,\mu}$ and $\gamma_1 = -\alpha\rho\frac{n-\mu}{2q}$. Similarly, in view of (4.23), the integral $I_2(t)$ can be estimated as

$$I_2(t) \le (2^{\rho+1}\varepsilon^{\rho})C\int_0^t (t-s)^{\gamma_2}\int_0^s R_{\alpha-1}(s-\tau)\tau^{-\beta\rho}\Sigma_2(\tau)d\tau ds, \tag{4.32}$$

where $\Sigma_2(\tau) = \|u(\tau) - v(\tau)\|_{p,\mu}$ and $\gamma_2 = \rho\beta - \alpha$.

Now setting $\Sigma(\tau) = \Sigma_1(\tau) + \Sigma_2(\tau)$ and making the changes $\tau = sz$ and $s = t\sigma$ in (4.31) and (4.32), we get

$$\begin{aligned} I_1(t) + I_2(t) \le (2^{\rho+1}\varepsilon^{\rho})C\int_0^1 (1-\sigma)^{\gamma_1}\sigma^{\alpha-1-\beta(\rho+1)}\int_0^1 R_{\alpha-1}(1-z)z^{-\beta(\rho+1)}\Sigma(t\sigma z)dzd\sigma + \\ +(2^{\rho+1}\varepsilon^{\rho})C\int_0^1 (1-\sigma)^{\gamma_2}\sigma^{\alpha-1-\beta\rho}\int_0^1 R_{\alpha-1}(1-z)z^{-\beta\rho}\Sigma(t\sigma z)dzd\sigma. \end{aligned} \tag{4.33}$$

Notice that $\limsup_{t\to+\infty}\Sigma(t) < \infty$ because $u, v \in H_q$. We claim that

$$\Pi := \limsup_{t\to+\infty}\Sigma(t) = 0, \tag{4.34}$$

which is equivalent to (3.11). To see this, we take $\limsup_{t\to+\infty}$ in (4.33) to get

$$\begin{aligned} \limsup_{t\to+\infty}[I_1(t) + I_2(t)] &\le (2^{\rho+1}\varepsilon^{\rho})C\int_0^1 (1-\sigma)^{\gamma_1}\sigma^{\alpha-1-\beta(\rho+1)}d\sigma \limsup_{t\to+\infty}\left(\int_0^1 R_{\alpha-1}(1-z)z^{-\beta(\rho+1)}\Sigma(t\sigma z)dz\right) \\ &+ (2^{\rho+1}\varepsilon^{\rho})C\int_0^1 (1-\sigma)^{\gamma_2}\sigma^{\alpha-1-\beta\rho}d\sigma \limsup_{t\to+\infty}\left(\int_0^1 R_{\alpha-1}(1-z)z^{-\beta\rho}\Sigma(t\sigma z)dz\right) \\ &\le (2^{\rho+1}\varepsilon^{\rho})C\left(\int_0^1 (1-\sigma)^{\gamma_1}\sigma^{\alpha-1-\beta(\rho+1)}d\sigma\int_0^1 R_{\alpha-1}(1-z)z^{-\beta(\rho+1)}dz\right)\Pi \\ &+ (2^{\rho+1}\varepsilon^{\rho})C\left(\int_0^1 (1-\sigma)^{\gamma_2}\sigma^{\alpha-1-\beta\rho}d\sigma\int_0^1 R_{\alpha-1}(1-z)z^{-\beta\rho}dz\right)\Pi \\ &= (K_1 + K_2)(2^{\rho+1}\varepsilon^{\rho})\Pi. \end{aligned} \tag{4.35}$$

Thanks to the inequalities (4.29), (4.30), (4.35) and the hypothesis (3.12), we obtain

$$\begin{aligned} \Pi &\le \limsup_{t\to+\infty}(t^{\beta}\|G_\alpha(t)(u_0 - v_0)\|_{q,\mu} + \|G_\alpha(t)(u_0 - v_0)\|_{p,\mu}) + \limsup_{t\to+\infty}[I_1(t) + I_2(t)] \\ &\le 0 + (K_1 + K_2)(2^{\rho+1}\varepsilon^{\rho})\Pi \\ &= (2^{\rho+1}\varepsilon^{\rho}K)\Pi \end{aligned} \tag{4.36}$$

which leads us to $\Pi = 0$, because $2^{\rho+1}\varepsilon^{\rho}K < 1$.

■

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