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On the Navier–Stokes equations in the half-space with initial and boundary rough data in Morrey spaces

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ABSTRACT

We consider the initial–boundary value problem for the Navier–Stokes equations in the half-space with data in Morrey spaces. Existence of small global solutions is proved in spaces with the right homogeneity to allow self-similar solutions. Moreover, we analyze the long time behavior of the solutions and obtain a class of asymptotically self-similar ones.

1. Introduction

We consider the initial–boundary value problem (IBVP) for the Navier–Stokes equations in the half-space $\mathbb{R}^n_+$ with $n \geqslant 3$, which reads as follows:

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$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \mathbb{R}^n_+, \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^n_+, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n_+, \tag{1.3}$$

$$u(x,t) = a(x,t) \quad \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \tag{1.4}$$

where the field $u = (u_1(x,t), \ldots, u_n(x,t))$ is the velocity of the fluid, the function $p(x,t)$ is the pressure, and $(u \cdot \nabla) = \sum_{j=1}^n u_j(\frac{\partial}{\partial x_j})$. The data $u_0(x) = (u_{01}(x), \ldots, u_{0n}(x))$ and $a(x,t) = (a_1(x,t), a_2(x,t), \ldots, a_n(x,t))$ stand for the initial and boundary values of the field u, respectively. Throughout this paper, spaces of scalar and vector valued functions are abusively denoted in the same way.

The existence of mild-type solutions for Navier–Stokes equations in the whole space $\mathbb{R}^n$ have been addressed by several authors, see e.g. [6,4,7,9,11–13,15,24,28] and references therein. In these works, the reader can find existence results of small global mild solutions with rough initial data in critical spaces such as: Lebesgue space L^n, Marcinkiewicz space $L^{n,\infty}$, homogeneous Besov space $\dot{B}^{n/q-1}_{q,\infty}$, pseudomeasure space $\mathcal{PM}^{n-1}$, Morrey space $\mathcal{M}_{p,n-p}$, BMO^{-1}, and Besov–Morrey space $\mathcal{N}^{n/q-1}_{q,\lambda,\infty}$ with $q \in (n,\infty)$.

From the point of view of critical spaces and semigroup theory, the case of the half-space domain $\mathbb{R}^n_+$ seems to be more difficult-to-treating than $\mathbb{R}^n$. The basic reasons are the non-compact boundary, the Stokes semigroup is not a convolution operator and derivatives do not commute with the Stokes semigroup and Leray–Helmholtz projector. Moreover, one needs to obtain local versions of spaces good-to-handling and, many times, to estimate boundary and trace operators in rough spaces. In comparison with $\mathbb{R}^n$, there are fewer existence results available in the literature.

Let us first review works about (1.1)–(1.4) dealing with homogeneous boundary condition ($a = 0$). So far, existence results of global mild solution for (1.1)–(1.4) have only been proved in the following critical spaces $L^n(\mathbb{R}^n_+)$ [14,25,27], $L^{n,\infty}(\mathbb{R}^n_+)$ [28] and the homogeneous Besov space $\dot{B}^{n/q-1}_{q,\infty}(\mathbb{R}^n_+)$ [3]. In [19], Saal studied (1.1)–(1.3) with a Robin boundary condition and obtained global solutions for data $u_0 \in L^n(\mathbb{R}^n_+)$. Existence results in L^∞ for the Stokes problem (linear case) have been obtained by [5,22,18] with Dirichlet and Robin boundary conditions, respectively. Solonnikov [22] also proved local-in-time existence for (1.1)–(1.4) with bounded and continuous initial data, which is (in general) nondecreasing at infinity. We also mention the papers [8] and [23] (see also their references), where the reader can found results on gradient estimates for the Stokes operator in Hardy spaces and $L^1(\mathbb{R}^n_+)$, based on Ukai's formulas.

The case with rough data $a \neq 0$ is particularly challenging and presents further difficulties such as:

- It is not possible to use a Helmholtz type decomposition and define a suitable Leray–Helmholtz projector;
- In general, one is not able to extend $a \neq 0$ to the interior of the domain and to reduce (by subtraction) the problem to the corresponding homogeneous case $a = 0$;
- A semigroup approach based in a resolvent analysis is very difficult to employ. Indeed, this seems not to be possible of performing.

In [16, Theorem 17], Lewis considered (1.1)–(1.4) with $a \neq 0$ and proved global existence of solutions in mixed Lebesgue spaces $L^p(0,\infty; L^q)$ by taking small data $u_0 \in L^{r_1} \cap L^{r_2} \subset L^n(\mathbb{R}^n_+)$ and $a \in L^d(0,\infty; L^r)$, where $r_1, r_2, p, q, r, d \neq \infty$, $r_1 < n < r_2$, $\frac{n-1}{r} + \frac{2}{d} = 1$ and $\frac{n}{q} + \frac{2}{p} = 1$. Indeed, he still assumed further restrictions on these exponents and on the data u_0, a. Assuming $a_n = 0$ and using the same formulation of [16], the author of [26] employed Besov spaces and showed existence of global self-similar solution with both $u_0 \in \dot{B}^{-1/2}_{6,\infty}(\mathbb{R}^3_+) \cap \dot{B}^{-1/4}_{4,\infty}(\mathbb{R}^3_+)$ and $\sup_{t>0} t^{1/3} \|a(x,t)\|_{L^6(\mathbb{R}^2)}$ small enough.

In this paper we prove existence, self-similar symmetry and asymptotic behavior of global solutions for the IBVP (1.1)–(1.4) in the framework of Morrey spaces. The initial and boundary data are both assumed to belong to such spaces, which contain functions that may be strongly rough and nondecreasing as $|x| \to \infty$. More precisely, we consider small initial data u_0 belonging to the critical Morrey space $\mathcal{M}_{p,n-p}(\mathbb{R}^n_+)$ and boundary data a as in (4.4). Let us recall that the inclusions

$$L^n(\mathbb{R}^n_+) \subset L^{n,\infty}(\mathbb{R}^n_+) \subset \mathcal{M}_{p,n-p}(\mathbb{R}^n_+)$$

hold true. Also, there is no any inclusion relation between the critical spaces $\mathcal{M}_{p,n-p}$ and $\dot{B}^{n/q-1}_{q,\infty}(\mathbb{R}^n_+)$, for $q > n$. Thus, in comparison to previous works, our existence results provide a new class of initial data, even for the homogeneous case $a = 0$.

The half-space $\mathbb{R}^n_+$ is invariant by the homothety $x \to \lambda x$, for every $\lambda > 0$. So, it makes sense the scaling

$$u(x,t) \to u_\lambda(x,t) = \lambda u\left(\lambda x, \lambda^2 t\right) \tag{1.5}$$

and the concept of self-similar solutions in $\mathbb{R}^n_+$ (invariant by (1.5)). This kind of symmetry is obtained when the boundary data satisfies $a(x,t) = \lambda a(\lambda x, \lambda^2 t)$, for all $\lambda > 0$, and u_0 is homogeneous of degree -1. We employ a suitable integral formulation for (1.1)–(1.4) found in [16] (see (3.6)). This equation is based on Green functions and Solonnikov formulas, while in [3,14,25,27,28] the authors have used semigroup-mild formulations or Ukai's formulas. Notice that $\mathbb{P}$ in (3.6) is the Leray–Helmholtz projector in $\mathbb{R}^n$ applied to extension $\bar{u}$ whose domain is $\mathbb{R}^n$. We analyze the integral formulation (3.6) considering $a(x,t)$ and $u(x,t)$ in time-weighted spaces *ala Kato* with indexes chosen for their norms to be scaling invariant. As a consequence we obtain existence of self-similar solutions (see Theorem 4.1).

In spirit of [2], we analyze the asymptotic behavior of solutions and obtain a class of asymptotically self-similar ones (see Theorem 4.2). In order to handle the integral formulation (3.6), we need to prove linear estimates for certain extension, boundary and trace operators connected to the structure of (3.6) (see Lemmas 3.1, 5.1, 5.2, 5.3). These three last lemmas seem to have an interest of its own. The arising difficulties are naturally transferred to the proof of Theorem 4.2, because the arguments used in some parts depend on those of the proof of existence theorem. Thus, the adaptation of the stability arguments of [2] involves certain care and it is not straightforwardly performed.

The plan of this paper is the following. In the next section we summarize some basic definitions and properties on Morrey spaces. The integral formulation for the IBVP (1.1)–(1.4) that we deal with is described in Section 3. In Section 4 we define suitable time-functional spaces and state our results, which are proved in Section 5.

2. Preliminaries

In this section we summarize some basic properties of Morrey spaces that we will need in this work. For a deeper discussion on these spaces, see [11,24,17].

Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with tangential component $x' = (x_1, \ldots, x_{n-1})$. In this paper, we use the abusive notation $f(x) = f(x', x_n)$ for functions f defined on $\mathbb{R}^n_+$ or $\mathbb{R}^n$. Henceforth, we consider either $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$.

For $1 \leqslant p \leqslant \infty$ and $0 \leqslant \mu \leqslant n$, the Morrey space $\mathcal{M}_{p,\mu} = \mathcal{M}_{p,\mu}(\Omega)$ is the space of all measurable functions such that

$$\|f\|_{\mathcal{M}_{p,\mu}(\Omega)} = \sup_{x_0 \in \Omega,\, r>0} r^{-\frac{\mu}{p}} \|f\|_{L^p(\Omega_r(x_0))} < \infty, \tag{2.1}$$

where $\Omega_r(x_0) = \{x \in \Omega;\ |x - x_0| < r\} \subset \Omega$ is the intersection between Ω and the closed ball in $\mathbb{R}^n$ with center x_0 and radius r. The space $\mathcal{M}_{p,\mu}$ endowed with the norm $\|\cdot\|_{\mathcal{M}_{p,\mu}(\Omega)}$ is a Banach space and $\mathcal{M}_{p,0} = L^p$ for $p > 1$. With the suitable interpretation of the integral in (2.1), $\mathcal{M}_{1,0}$ coincides with the space of finite Radon measures on Ω, and $L^\infty(\Omega) = \mathcal{M}_{\infty,\mu}(\Omega) = \mathcal{M}_{p,n}(\Omega)$.

We also define the weak Morrey space $\mathcal{M}_{p,\infty,\lambda}$ as the Banach space of measurable functions whose norm is given by

$$\|f\|_{\mathcal{M}_{p,\infty,\mu}(\Omega)} = \sup_{x_0\in\Omega,\, r>0} r^{-\frac{\mu}{p}} \|f\|_{L^{p,\infty}(\Omega_r(x_0))} < \infty, \tag{2.2}$$

where $L^{p,\infty}$ denotes the weak-L^p space.

Define $C^k_{0,\sigma}(\Omega)$ and $\mathcal{M}^{\sigma}_{q,\mu}(\Omega)$ to be the set of all vectors $u : \Omega \to \mathbb{R}^n$ such that $\nabla \cdot u = 0$ and u_i belongs to $C^k_0(\Omega)$ and $\mathcal{M}_{q,\mu}(\Omega)$, respectively, for all $i = 1, \ldots, n$. Here $C^k_0(\Omega)$ is the set of compact supported functions with derivatives of order k continuous. Also, we denote $\|u\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} = \max_{i=1,\ldots,n} \|u_i\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}$. The letter C will stand for generic positive constants that may change from line to line or even within the same line.

In what follows we recall some useful inclusions and inequalities in Morrey spaces.

Lemma 2.1. *Let Ω be either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$. Assume that $1 \leqslant p_i \leqslant \infty$ and $0 \leqslant \mu_i < n$, for all $i = 1, 2, 3$.*

(i) (*Inclusion*) *Let $p_1 \leqslant p_2$ and $\frac{n-\mu_1}{p_1} = \frac{n-\mu_2}{p_2}$. Then*

$$\mathcal{M}_{p_2,\mu_2}(\Omega) \subset \mathcal{M}_{p_1,\mu_1}(\Omega). \tag{2.3}$$

(ii) (*Hölder Inequality*) *If $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$ and $\frac{\mu_3}{p_3} = \frac{\mu_2}{p_2} + \frac{\mu_1}{p_1}$ then*

$$\|h_1 h_2\|_{\mathcal{M}_{p_3,\mu_3}(\Omega)} \leqslant \|h_1\|_{\mathcal{M}_{p_1,\mu_1}(\Omega)} \|h_2\|_{\mathcal{M}_{p_2,\mu_2}(\Omega)}. \tag{2.4}$$

(iii) (*Homogeneous functions*) *Let $\varphi \in L^\infty(\mathbb{S}^{n-1} \cap \Omega)$, $0 < d < n$ and $1 \leqslant r < n/d$. Then $\varphi(x/|x|)|x|^{-d} \in \mathcal{M}_{r,n-rd}$.*

(iv) (*Convexity*) *Let $p_1 < p_3 < p_2$ and $\kappa \in (0, 1)$ be such that $\frac{1}{p_3} = \frac{1}{p_1}\kappa + \frac{1}{p_2}(1-\kappa)$ and $\frac{\mu_3}{p_3} = \frac{\mu_1}{p_1}\kappa + \frac{\mu_2}{p_2}(1-\kappa)$. Then*

$$\|f\|_{\mathcal{M}_{p_3,\mu_3}(\Omega)} \leqslant C\big(\|f\|_{\mathcal{M}_{p_1,\infty,\mu_1}(\Omega)}\big)^{\kappa} \big(\|f\|_{\mathcal{M}_{p_2,\infty,\mu_2}(\Omega)}\big)^{1-\kappa}. \tag{2.5}$$

Proof. The proofs of (i) and (ii) can found in [11, pp. 130–132] and a direct computation yields (iii). A version of the item (iv), with norms (2.1) in place of (2.2) on the right side of (2.5), has been proved for instance in [11, p. 132]. For the reader convenience, we prove (2.5). From interpolation properties (see [10, Proposition 1.1.14, p. 8]), one has $L^{p_3}(\Omega_r(x_0)) \subset L^{p_1,\infty}(\Omega_r(x_0)) \cap L^{p_2,\infty}(\Omega_r(x_0))$ with

$$\|f\|_{L^{p_3}(\Omega_r(x_0))} \leqslant C\|f\|^{\kappa}_{L^{p_1,\infty}(\Omega_r(x_0))} \|f\|^{1-\kappa}_{L^{p_2,\infty}(\Omega_r(x_0))}. \tag{2.6}$$

Now we obtain (2.5) by multiplying (2.6) by $r^{-\frac{\mu_3}{p_3}} = r^{-\frac{\mu_1}{p_1}\kappa} r^{-\frac{\mu_2}{p_2}(1-\kappa)}$ and afterwards taking the supremum over $r > 0$ and $x_0 \in \Omega$. $\square$

Notice that the last proof is somewhat different from that of [11], because one cannot apply Hölder inequality (even in Lorentz spaces) in order to get (2.6).

The Riesz potential $(-\Delta)^{-\frac{\delta}{2}}$ works well in Morrey spaces (see e.g. [24, Proposition 3.7]).

Lemma 2.2. *Let $0 < \delta < n$, $1 < p_1 < p_2 < \infty$, $0 \leqslant \mu < n$ be such that $\delta = \frac{n-\mu}{p_1} - \frac{n-\mu}{p_2}$. There exists $C > 0$ such that*

$$\left\| \frac{1}{|x|^{n-\delta}} * f \right\|_{\mathcal{M}_{p_2,\mu}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathcal{M}_{p_1,\mu}(\mathbb{R}^n)}, \tag{2.7}$$

for all $f \in \mathcal{M}_{p_1,\mu}(\mathbb{R}^n)$.

We also recall a Sobolev trace type inequality in Morrey spaces (see [1, Theorem 5.1]).

Lemma 2.3. *Let* $0 < \delta < n$, $0 \leqslant \mu < n-1$ *and* $1 \leqslant q_1 < q_2 < \infty$ *be such that* $\frac{1}{\delta} < q_1 < \frac{(n-\mu)}{\delta}$ *and* $\frac{n-1-\mu}{q_2} = \frac{n-\mu}{q_1} - \delta$. *Then*

$$\left\| f\left(x', 0\right)\right\|_{\mathcal{M}_{q_2,\infty,\mu}(\mathbb{R}^{n-1},dx')} \leqslant C\left\|\left[(-\Delta_x)^{\frac{\delta}{2}} f\right](x)\right\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^n,dx)}, \tag{2.8}$$

where $C > 0$ *is a constant independent of* f.

2.1. Heat semigroup in $\mathbb{R}^n$

The heat semigroup $\{G(t)\}_{t\geqslant 0}$ in $\mathbb{R}^n$ is the family of convolution operators defined by $G(t)\varphi = g(\cdot,t) * \varphi$, where $g(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ is the so-called Gaussian kernel. Note that

$$g(x,t) = \lambda^n g\left(\lambda x, \lambda^2 t\right), \quad \text{for all } \lambda > 0,\ t > 0 \text{ and } x \in \mathbb{R}^n. \tag{2.9}$$

Despite of the strong continuity at $t > 0$, the heat semigroup $\{G(t)\}_{t\geqslant 0}$ is only weak-continuous in $\mathcal{M}_{p,\mu}$ as $t \to 0^+$. The reason is that the identity approximation does not work well in $\mathcal{M}_{p,\mu}$ with $0 < \mu < n$, because it contains singular functions like the homogeneous ones.

In the sequel we recall an estimate for the heat semigroup in $\mathbb{R}^n$ found in [11, Lemma 2.1].

Lemma 2.4. *Let* $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, $0 \leqslant \mu < n$, $\gamma_1 = \frac{n-\mu}{q_1}$, $\gamma_2 = \frac{n-\mu}{q_2}$ *and let* k *be a multi-index. There exists* $C > 0$ *such that*

$$\left\|\nabla_x^k G(t) u_0\right\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^n)} \leqslant C t^{-\frac{1}{2}(\gamma_1-\gamma_2)-\frac{|k|}{2}} \|u_0\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^n)}, \tag{2.10}$$

for all $u_0 \in \mathcal{M}_{q_1,\mu}(\mathbb{R}^n)$ *and* $t > 0$. *The statement still holds true with* $(-\Delta_x)^{\frac{|k|}{2}}$ *in place of* ∇_x^k.

3. An integral formulation for (1.1)–(1.4)

Before stating an integral formulation for (1.1)–(1.4), we recall an extension to $\mathbb{R}^n$ of vector functions defined in $\mathbb{R}^n_+$. From [20, Lemma 1, p. 182], there is a continuous function $\psi : [1,\infty) \to \mathbb{R}$ satisfying

$$\psi(s) = O\left(s^{-N}\right) \quad \text{as } s \to \infty, \text{ for all } N, \tag{3.1}$$

and

$$\int_1^\infty \psi(s)\, ds = 1 \quad \text{and} \quad \int_1^\infty s^k \psi(s)\, ds = 0, \quad \text{for all } k \in \mathbb{N}.$$

Let $u = (u_1, u_2, \ldots, u_n) \in \mathcal{M}_{p,\mu}(\mathbb{R}^n_+)$ and define the extension $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$, for a.e. $x \in \mathbb{R}^n$, by

$$\bar{u}_i\big(x', x_n\big) = \begin{cases} u_i(x', x_n), & \text{if } x_n > 0, \\ \int_1^\infty (1-2s)u_i(x', (1-2s)x_n)\psi(s)\,ds, & \text{if } x_n \leqslant 0 \text{ and } i = 1, 2, \ldots, n-1, \\ \int_1^\infty u_i(x', (1-2s)x_n)\psi(s)\,ds, & \text{if } x_n \leqslant 0 \text{ and } i = n, \end{cases} \tag{3.2}$$

when the integrals in (3.2) converge.

Lemma 3.1. *Assume that $1 \leqslant p < \infty$, $0 \leqslant \mu < n$ and let $\bar{u}$ be as in (3.2). If $u \in \mathcal{M}_{p,\mu}(\mathbb{R}^n_+)$ and $div(u) = 0$ in $\mathcal{S}'(\mathbb{R}^n_+)$ then $\bar{u} \in \mathcal{M}_{p,\mu}(\mathbb{R}^n)$ and $div(\bar{u}) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, there exists $C > 0$ such that*

$$\|\bar{u}\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} \leqslant C\|u\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}, \tag{3.3}$$

for all $u \in \mathcal{M}_{p,\mu}(\mathbb{R}^n_+)$.

Proof. The change of variables $(x', (2s-1)x_n) \to (x', x_n)$ yields

$$(2s-1)\big\|u_i\big(x', (2s-1)x_n\big)\big\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)} \leqslant C(2s-1)^{1-\frac{1-\mu}{p}}\big\|u_i\big(x', x_n\big)\big\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}.$$

Thus, for $i = 1, 2, \ldots, n-1$, we have

$$\begin{aligned} \big\|\bar{u}_i\big(x', x_n\big)1_{\{x_n \leqslant 0\}}\big\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} &= \left\|\int_1^\infty (1-2s)u_i\big(x', (2s-1)x_n\big)\psi(s)\,ds\right\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)} \\ &\leqslant \int_1^\infty (2s-1)\big\|u_i\big(x', (2s-1)x_n\big)\big\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}\big|\psi(s)\big|\,ds \\ &\leqslant C\int_1^\infty (2s-1)^{1-\frac{1-\mu}{p}}\big|\psi(s)\big|\,ds\,\|u_i\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)} \\ &= C\|u_i\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}, \end{aligned} \tag{3.4}$$

because of (3.1) and $2s-1 \geqslant 1$ when $s \geqslant 1$. Similarly, one obtains

$$\big\|\bar{u}_n\big(x', x_n\big)1_{\{x_n \leqslant 0\}}\big\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} \leqslant C\int_1^\infty (2s-1)^{-\frac{1-\mu}{p}}\big|\psi(s)\big|\,ds\,\|u_n\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)} = C\|u_n\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}. \tag{3.5}$$

Since $\|\bar{u}1_{\{x_n>0\}}\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} = \|u\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}$, the estimate (3.3) follows from (3.4)–(3.5). In view of (3.2), a simple computation shows that $div(\bar{u}) = 0$ when $div(u) = 0$. $\square$

For the velocity field u, the problem (1.1)–(1.4) can be formally transformed into the following integral equation (see [16, p. 759]):

$$u = B(\bar{u}, \bar{u}) + \mathcal{K}\big[B(\bar{u}, \bar{u})|_0\big] + \mathcal{K}[a] + G(t)\bar{u}_0 - \mathcal{K}\big[\big(G(t)\bar{u}_0\big)\big|_0\big] \tag{3.6}$$

where

$$B(\bar{u},\bar{u})(x,t) = -\int_0^t G(t-s)\mathbb{P}(\bar{u}\cdot\nabla\bar{u})(s)\,ds = -\int_0^t \nabla\cdot G(t-s)\mathbb{P}(\bar{u}\otimes\bar{u})(s)\,ds, \tag{3.7}$$

the operator $\mathbb{P}$ is the Leray–Helmholtz projector in $\mathbb{R}^n$ and $\varphi(x)|_0 = \varphi(x',0)$ stands for the restriction of φ to $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Let $x = (x', x_n)$ and $f(x,t) = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))$. The above operator $\mathcal{K}$ can be regarded as (see [16] and [21, p. 36])

$$\mathcal{K}[f] = \left(\sum_{j=1}^n \mathcal{K}_{1,j}[f_j], \sum_{j=1}^n \mathcal{K}_{2,j}[f_j], \ldots, \sum_{j=1}^n \mathcal{K}_{n,j}[f_j]\right) + \mathcal{P}[f_n] \tag{3.8}$$

with

$$\mathcal{K}_{i,j}[\phi](x,t) = \mathcal{K}_{i,j}[\phi]\big(x', x_n, t\big) = \int_0^t \int_{\mathbb{R}^{n-1}} L_{i,j}\big(x'-y, x_n, t-s\big)\phi(y,s)\,dy\,ds, \tag{3.9}$$

for $i,j = 1,2,\ldots,n$, and $\mathcal{P}[f_n] = (\mathcal{P}_1[f_n], \mathcal{P}_2[f_n], \ldots, \mathcal{P}_n[f_n])$ where

$$\mathcal{P}_j[f_n](x,t) = \int_{\partial\mathbb{R}^n_+} P_j(x,y) f_n(y,t)\,dy. \tag{3.10}$$

The kernel P_j is given explicitly by

$$P_j(x,y) = -(n-2)\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\frac{\partial}{\partial x_j}\frac{1}{[(x_1-y_1)^2 + \cdots + (x_n-y_n)^2]^{\frac{n}{2}-1}}, \tag{3.11}$$

where $x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$, and $\Gamma(\cdot)$ stands for the gamma function. In particular, we have the estimate

$$\big|P_j(x,y)\big| \leqslant \frac{C}{(|x'-y|^2 + x_n^2)^{(n-1)/2}}. \tag{3.12}$$

The kernel $L_{i,j}(x', x_n, t) = L_{i,j}(x,t)$ in (3.9) is given by

$$L_{i,j}(x,t) = -2\delta_{ij}\frac{\partial g}{\partial x_n}(x,t) - 2\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n-1}}\int_0^{x_n}\frac{\partial g}{\partial y_n}(y,t)\frac{y_i - x_i}{|y-x|^n}\,dy_n\,dy' \tag{3.13}$$

and satisfies (see [21, Theorem 3, p. 41])

$$L_{i,j}\big(\lambda x, \lambda^2 t\big) = \lambda^{-n-1}L_{i,j}(x,t), \quad \text{for all } \lambda > 0,\ t > 0,\ x \in \mathbb{R}^n_+, \tag{3.14}$$

and

$$\big|L_{i,j}(x,t)\big| \leqslant \frac{C}{t^{1/2}(|x|^2+t)^{n/2}} = \frac{C}{t^{1/2}(|x'|^2 + x_n^2 + t)^{n/2}}, \tag{3.15}$$

for all $t > 0$, $x \in \mathbb{R}^n_+$, and $i, j = 1, 2, \ldots, n$. Taking $\lambda = t^{-1/2}$ in (3.14), it follows that

$$L_{i,j}(x,t) = t^{-\frac{n+1}{2}} L_{i,j}\big(t^{-1/2}x, 1\big), \quad \text{for all } t > 0,\ x \in \mathbb{R}^n_+. \tag{3.16}$$

One can write the integral equation (3.6) in the following shortly way:

$$u = \mathcal{N}(u,u) + \mathcal{K}[a] + \mathcal{L}(u_0), \tag{3.17}$$

where

$$\mathcal{N}(u,u) = B(\bar{u},\bar{u}) + \mathcal{K}\big[B(\bar{u},\bar{u})|_0\big] \quad \text{and} \quad \mathcal{L}(u_0) = G(t)\bar{u}_0 - \mathcal{K}\big[\big(G(t)\bar{u}_0\big)\big|_0\big]. \tag{3.18}$$

Remark 3.1. The pressure p can be formally recovered by applying *div* in (1.1) and solving the resulting equation for p in the sense of distributions.

4. Functional setting and results

In this section we state our results for the IBVP (1.1)–(1.4) in Morrey spaces. Before stating them, we give some notations and define suitable time-dependent function spaces where (1.1)–(1.4) will be handled.

The Navier–Stokes equation (1.1) presents the following scaling for u

$$u(x,t) \to u_\lambda(x,t) = \lambda u\big(\lambda x, \lambda^2 t\big). \tag{4.1}$$

The map (4.1) induces in a natural way a scaling for the initial data $u_0(x)$ and boundary data $a(x,t)$, namely

$$u_0(x) \to u_{0,\lambda}(x) = \lambda u_0(\lambda x) \quad \text{and} \quad a(x,t) \to a_\lambda(x,t) = \lambda a\big(\lambda x, \lambda^2 t\big). \tag{4.2}$$

One of our aim is to obtain existence of self-similar solutions in Morrey spaces for (1.1)–(1.4), that is, solutions invariant by (4.1). For that matter, we need to study (1.1)–(1.4) in function spaces whose norms are invariant by (4.1) and (4.2). Recall that the notation $BC((0,\infty), Y)$ stand for the class of bounded continuous functions from interval $(0,\infty)$ into the Banach Y.

Let $2 < p, q < \infty$, $1 < r < \infty$, $\mu = n - p$, $\alpha = \frac{1}{2} - \frac{n-\mu}{2q}$, and $\beta = \frac{1}{2} - \frac{n-1-\mu}{2r}$. We define the Banach space

$$H_q = \big\{u \text{ is measurable; } t^\alpha u(x,t) \in BC\big((0,\infty), \mathcal{M}^\sigma_{q,\mu}\big(\mathbb{R}^n_+\big)\big)\big\},$$

with norm given by

$$\|u\|_{H_q} = \sup_{t>0} t^\alpha \big\|u(\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}. \tag{4.3}$$

Let us also define the boundary-data space $\widetilde{H}_r$ by the set of measurable vectors $a = (a_1, \ldots, a_n)$ such that

$$t^\beta a(x,t) \in BC\big((0,\infty), \mathcal{M}_{r,\mu}\big(\mathbb{R}^{n-1}\big)\big) \quad \text{and}$$
$$t^\alpha a_n(x,t) \in BC\big((0,\infty), \mathcal{M}_{(p-1)q/p,\mu}\big(\mathbb{R}^{n-1}\big)\big) \tag{4.4}$$

which is a Banach space with norm

$$\|a\|_{\widetilde{H}_r} = \sup_{t>0} t^{\beta} \left\| a(\cdot,t) \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} + \sup_{t>0} t^{\alpha} \left\| a_n(\cdot,t) \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}. \tag{4.5}$$

Notice that (4.3) and (4.5) are invariant by (4.1) and (4.2), respectively.

From now on, a solution for (3.6), or equivalently (3.17), will be called a *mild solution* for IBVP (1.1)–(1.4). Now we can state our existence result.

Theorem 4.1. *Let $2 < p < q\frac{p-1}{p} < r < q < \infty$, $\mu = n - p \geqslant 0$ and $u_0 \in \mathcal{M}^{\sigma}_{p,\mu}(\mathbb{R}^n_+)$.*

- (i) (*Existence and uniqueness*) *There exists $\varepsilon > 0$ and $\delta = \delta(\varepsilon)$ ($\delta = C\varepsilon$) such that, if $\|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)} \leqslant \delta$ and $\|a\|_{\widetilde{H}_r} \leqslant \delta$ then the IBVP (1.1)–(1.4) has a global mild solution $u \in H_q$, which is unique one in the closed ball $\{u \in H_q;\ \|u\|_{H_q} \leqslant 2\varepsilon\}$. The data-solution map $[u_0, a] \to u$ is locally Lipschitz continuous.*
- (ii) (*Self-similarity*) *Assume that $u_0(x)$ and $a(x,t)$ are invariant by (4.2), that is, u_0 is homogeneous of degree -1 and $a(x,t) = \lambda a(\lambda x, \lambda^2 t)$ for all $\lambda > 0$, $t > 0$ and a.e. $x \in \mathbb{R}^n_+$. Then the solution obtained through item* (i) *is self-similar, that is, $u(x,t) = \lambda u(\lambda x, \lambda^2 t)$, for all $\lambda > 0$, $t > 0$ and $x \in \mathbb{R}^n_+$.*

We also prove an asymptotic stability result which shows that certain perturbations of the initial data $u_0(x)$ and boundary data $a(x,t)$ go to zero as $t \to +\infty$. In particular, it implies the existence of an attractor-basin around each self-similar solution.

Theorem 4.2 *(Asymptotic stability). Assume the hypotheses of Theorem 4.1. Recall that w_n denotes the n-th coordinate of a vector w. Let u and v be mild solutions given by Theorem 4.1 and corresponding to the boundary and initial data a, u_0 and b, v_0, respectively. If*

$$\lim_{t\to\infty} t^{\alpha} \left\| G(t)(\bar{u}_0 - \bar{v}_0) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} = \lim_{t\to\infty} t^{\alpha} \left\| \left(\left[G(t)(\bar{u}_0 - \bar{v}_0) \right]_n \right) \big|_0 \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} = 0, \tag{4.6}$$

$$\lim_{t\to\infty} t^{\beta} \left\| \left[G(t)(\bar{u}_0 - \bar{v}_0) \right] \big|_0 \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} = 0, \tag{4.7}$$

and

$$\lim_{t\to\infty} t^{\alpha} \left\| a_n(\cdot,t) - b_n(\cdot,t) \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} = \lim_{t\to\infty} t^{\beta} \left\| a(\cdot,t) - b(\cdot,t) \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} = 0, \tag{4.8}$$

then

$$\lim_{t\to\infty} t^{\alpha} \left\| u(\cdot,t) - v(\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} = 0. \tag{4.9}$$

Remark 4.1 *(Basin of attraction).* Let u be a self-similar solution with data $u_0(x)$ and $a(x,t)$. Take $v_0(x) = u_0(x) + \varphi(x)$ and $b(x,t) = a(x,t) + \psi(x,t)$ with $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ being smooth compacted small vector functions. Consider the mild solution v with data $v_0(x)$ and $b(x,t)$. Then the perturbed mild solution v is attracted to the self-similar solution u in the sense of (4.9), because we have

$$\begin{aligned}
\lim_{t\to\infty} t^{\alpha} \left\| G(t)\bar{\varphi} \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} &= \lim_{t\to\infty} t^{\alpha} \left\| \left(G(t)\bar{\varphi} \right)_n \big|_0 \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} \\
&= \lim_{t\to\infty} t^{\beta} \left\| \left(G(t)\bar{\varphi} \right) \big|_0 \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} = 0, \\
\lim_{t\to\infty} t^{\alpha} \left\| \psi_n(\cdot,t) \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} &= \lim_{t\to\infty} t^{\beta} \left\| \psi(\cdot,t) \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} = 0.
\end{aligned}$$

5. Proof of results

In this section we give the proofs of the results stated previously. We start by providing key estimates on Morrey spaces for some linear operators appearing within integral equation (3.6).

5.1. Linear estimates

In the following lemma we prove estimates for the boundary integral operator $\mathcal{K}$ defined in (3.8).

Lemma 5.1. *Let $1 < q_1 < q_2 < \infty$, $0 \leqslant \mu < n-1$ be such that*

$$\delta = \frac{n-1-\mu}{q_1} - \frac{n-1-\mu}{q_2} < \frac{1}{q_2}, \tag{5.1}$$

and let $\eta < \frac{2+\delta}{2} - \frac{1}{2q_2}$. There exists a positive constant $C > 0$ (independent of ϕ) such that

$$\sup_{t>0} t^{\eta} \big\| \mathcal{K}_{i,j}[\phi](\cdot,t) \big\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^n_+)} \leqslant C \sup_{t>0} t^{\eta - (\frac{n-1-\mu}{2q_1} - \frac{n-\mu}{2q_2})} \big\| \phi(\cdot,t) \big\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}, \tag{5.2}$$

for all $i, j = 1, 2, \ldots, n$.

Proof. Recall the notation $\Omega_r(x_0) = \{x \in \mathbb{R}^n_+;\ |x - x_0| < r\}$ and let $[\Omega_r(x_0)]'$ stand for the projection of $\Omega_r(x_0)$ onto $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+:\ x_n = 0\}$. Since $\Omega_r(x_0) \subset [\Omega_r(x_0)]' \times (0,\infty)$, we have that

$$\begin{aligned} \big\| \mathcal{K}_{i,j}[\phi](x,t) \big\|_{L^{q_2}(\Omega_r(x_0))} &\leqslant \big\| \mathcal{K}_{i,j}[\phi]\big(x', x_n, t\big) \big\|_{L^{q_2}([\Omega_r(x_0)]' \times (0,\infty))} \\ &= \big\| \big\| \mathcal{K}_{i,j}[\phi]\big(x', x_n, t\big) \big\|_{L^{q_2}((0,\infty),dx_n)} \big\|_{L^{q_2}([\Omega_r(x_0)]',dx')}, \end{aligned} \tag{5.3}$$

for each fixed $t > 0$. Take $0 < \theta < 1$ in such a way that $n - 1 - \delta = (1-\theta)(n - \frac{1}{q_2})$. Using Minkowski inequality for integrals and (3.15), we estimate

$$\begin{aligned} \big\| \mathcal{K}_{i,j}[\phi]\big(x', x_n, t\big) \big\|_{L^{q_2}((0,\infty),dx_n)} &\leqslant \int_0^t \int_{\mathbb{R}^{n-1}} \big\| L_{i,j}\big(x'-y, x_n, t-s\big)\phi(y,s) \big\|_{L^{q_2}((0,\infty),dx_n)}\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^{n-1}} \big\| L_{i,j}\big(x'-y, x_n, t-s\big) \big\|_{L^{q_2}((0,\infty),dx_n)} \big|\phi(y,s)\big|\, dy\, ds \\ &\leqslant C \int_0^t \int_{\mathbb{R}^{n-1}} \frac{|\phi(y,s)|}{(t-s)^{1/2}(|x'-y|^2 + |t-s|)^{\frac{n}{2} - \frac{1}{2q_2}}}\, dy\, ds \qquad (5.4) \\ &\leqslant C \int_0^t (t-s)^{-\frac{1}{2} - \frac{\theta}{2}(n - \frac{1}{q_2})} \left[\int_{\mathbb{R}^{n-1}} \frac{|\phi(y,s)|}{|x'-y|^{(1-\theta)(n - \frac{1}{q_2})}}\, dy \right] ds, \qquad (5.5) \end{aligned}$$

where we have used $(z_1 + z_2)^{-k} \leqslant (z_1)^{-k\theta}(z_2)^{-k(1-\theta)}$ in passing from (5.4) to (5.5). It follows from definition (2.1) and estimate (2.7) that

$$r^{-\frac{\mu}{q_2}}\left\| \int\limits_{\mathbb{R}^{n-1}} \frac{|\phi(y,s)|}{|x'-y|^{(1-\theta)(n-\frac{1}{q_2})}}\,dy \right\|_{L^{q_2}([\Omega_r(x_0)]',dx')} \leqslant \left\| \int\limits_{\mathbb{R}^{n-1}} \frac{|\phi(y,s)|}{|x'-y|^{(1-\theta)(n-\frac{1}{q_2})}}\,dy \right\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^{n-1})}$$
$$\leqslant C\big\|\phi(\cdot,s)\big\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}, \tag{5.6}$$

for all $r>0$. Notice that the hypotheses on parameters imply $d=\eta-(\frac{n-1-\mu}{2q_1}-\frac{n-\mu}{2q_2})<1$ and $\frac{1}{2}+\frac{\theta}{2}(n-\frac{1}{q_2})<1$. Now we use (5.3), (5.5) and (5.6) to obtain

$$r^{-\frac{\mu}{q_2}}\big\|\mathcal{K}_{i,j}[\phi](x,t)\big\|_{L^{q_2}(\Omega_r(x_0))} \leqslant r^{-\frac{\mu}{q_2}}\big\|\big\|\mathcal{K}_{i,j}[\phi](x,t)\big\|_{L^{q_2}((0,\infty),dx_n)}\big\|_{L^{q_2}([\Omega_r(x_0)]',dx')}$$
$$\leqslant C\int\limits_0^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q_2})}\big\|\phi(\cdot,s)\big\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}\,ds \tag{5.7}$$
$$\leqslant C\int\limits_0^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q_2})}s^{-d}\,ds\Big[\sup_{s>0} s^d\big\|\phi(\cdot,s)\big\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}\Big]$$
$$= Ct^{\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q_2})-d}\Big[\sup_{s>0} s^d\big\|\phi(\cdot,s)\big\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}\Big]. \tag{5.8}$$

Taking in (5.8) the supremum over all $r>0$ and $x_0\in\mathbb{R}^n_+$, and afterwards over $t>0$, we deduce (5.2), because

$$\frac{1}{2}-\frac{\theta}{2}\left(n-\frac{1}{q_2}\right)-d=\frac{1}{2}+\frac{-1-\delta+\frac{1}{q_2}}{2}-d$$
$$=-\frac{\delta}{2}+\frac{1}{2q_2}+\frac{n-1-\mu}{2q_1}-\frac{n-\mu}{2q_2}-\eta=-\eta. \qquad \square$$

In the sequel we derive estimates for the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ defined by

$$\mathcal{F}_1[f](x',t)=\int\limits_0^t\int\limits_{\mathbb{R}^n}[\nabla_x\cdot g](x'-z',z_n,t-s)f(z',z_n,s)\,dz_n\,dz'\,ds, \tag{5.9}$$

$$\mathcal{F}_2[\varphi](x',t)=\big(G(t)\varphi\big)\big|_0=\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{R}^{n-1}} g(x'-z',z_n,t)\varphi(z',z_n)\,dz'\,dz_n, \tag{5.10}$$

where $g(x,t)=(4\pi t)^{-n/2}e^{-|x|^2/4t}$ is the heat kernel in $\mathbb{R}^n$.

Lemma 5.2. *Let $0\leqslant\eta<1/2$, $0\leqslant\mu<n-1$ and $1<d_1<d_2\leqslant\infty$ be such that $\frac{n-\mu}{d_1}-\frac{n-1-\mu}{d_2}<1$. There exists $C>0$ (independent of f and φ) such that*

$$\big\|\mathcal{F}_1[f](\cdot,t)\big\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})}\leqslant Ct^{-(\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2}+2\eta-\frac{1}{2})}\sup_{t>0}t^{2\eta}\big\|f(\cdot,t)\big\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}, \tag{5.11}$$

$$\big\|\mathcal{F}_2[\varphi](\cdot,t)\big\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})}\leqslant Ct^{-(\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2})}\|\varphi\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}, \tag{5.12}$$

for all $t>0$.

Proof. We only give the proof of (5.11) because (5.12) follows by similar arguments. Since g is a radial function and in view of (2.9), we have

$$\begin{aligned} \left|\mathcal{F}_1[f]\left(x',t\right)\right| &\leqslant \int_0^t \int_{\mathbb{R}^n} \tilde{g}\left(x'-z',z_n,t-s\right)\left|f\left(z',z_n,s\right)\right| dz'\,dz_n\,ds \\ &:= \int_0^t h\left(x',0,t,s\right) ds, \end{aligned} \tag{5.13}$$

where $\tilde{g} = |\nabla_x \cdot g|$ is a positive radial function satisfying $\tilde{g}(\lambda x', \lambda x_n, \lambda^2 t) = \lambda^{-(1+n)}\tilde{g}(x', x_n, t)$ and

$$h(x,t,s) = h\left(x',x_n,t,s\right) = \int_{\mathbb{R}^n} \tilde{g}\left(x'-z',x_n-z_n,t-s\right)\left|f\left(z',z_n,s\right)\right| dz'\,dz_n. \tag{5.14}$$

In the following we estimate the integral (5.13) in three steps, where the first two ones will used to handle (5.14).

First step (*case* $d_2 < \infty$). Let $0 < \delta < 1$, d_3, d_4 and $\kappa \in (0,1)$ be such that

$$1 < d_1 < \frac{1}{\delta} < d_3 < d_2 < d_4 < \infty \quad \text{and} \quad \frac{n-1-\mu}{d_2} = \frac{n-1-\mu}{d_3}\kappa + \frac{n-1-\mu}{d_4}(1-\kappa). \tag{5.15}$$

Choose $\frac{1}{\delta} < d_5, d_6 < \frac{n-\mu}{\delta}$ in a way that $d_1 < d_5 < d_3$, $d_1 < d_6 < d_4$ and

$$\frac{n-1-\mu}{d_3} = \frac{n-\mu}{d_5} - \delta \quad \text{and} \quad \frac{n-1-\mu}{d_4} = \frac{n-\mu}{d_6} - \delta. \tag{5.16}$$

Now, we can use (2.5) and afterwards the trace inequality (2.8) to estimate

$$\begin{aligned} \left\|h\left(x',0,t,s\right)\right\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})} &\leqslant C\left\|h\left(x',0,t,s\right)\right\|^{\kappa}_{\mathcal{M}_{d_3,\infty,\mu}(\mathbb{R}^{n-1})}\left\|h\left(x',0,t,s\right)\right\|^{1-\kappa}_{\mathcal{M}_{d_4,\infty,\mu}(\mathbb{R}^{n-1})} \\ &\leqslant C\left\|(-\Delta_x)^{\frac{\delta}{2}} h(x,t,s)\right\|^{\kappa}_{\mathcal{M}_{d_5,\mu}(\mathbb{R}^n)}\left\|(-\Delta_x)^{\frac{\delta}{2}} h(x,t,s)\right\|^{1-\kappa}_{\mathcal{M}_{d_6,\mu}(\mathbb{R}^n)}. \end{aligned} \tag{5.17}$$

From Lemma 2.4, we have that

$$\begin{aligned} \left\|(-\Delta_x)^{\frac{\delta}{2}} h(x,t,s)\right\|_{\mathcal{M}_{d_5,\mu}(\mathbb{R}^n)} &\leqslant \left\|\int_{\mathbb{R}^n} \left[(-\Delta_x)^{\frac{\delta}{2}}\tilde{g}\right](x-z,t-s)\left|f\left(z',z_n,s\right)\right| dz\,ds\right\|_{\mathcal{M}_{d_5,\mu}(\mathbb{R}^n)} \\ &\leqslant C(t-s)^{-\frac{1+\delta}{2}-(\frac{n-\mu}{2d_1}-\frac{n-\mu}{2d_5})}\left\|f(\cdot,s)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)} \end{aligned} \tag{5.18}$$

and similarly,

$$\left\|(-\Delta_x)^{\frac{\delta}{2}} h(x,t,s)\right\|_{\mathcal{M}_{d_6,\mu}(\mathbb{R}^n)} \leqslant C(t-s)^{-\frac{1+\delta}{2}-(\frac{n-\mu}{2d_1}-\frac{n-\mu}{2d_6})}\left\|f(\cdot,s)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}. \tag{5.19}$$

Inserting (5.18)–(5.19) into (5.17) and using (5.15)–(5.16), we obtain

$$\begin{aligned}\big\|h\big(x',0,t,s\big)\big\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})} &\leqslant C(t-s)^{-\kappa[\frac{1+\delta}{2}+(\frac{n-\mu}{2d_1}-\frac{n-\mu}{2d_5})]-(1-\kappa)[\frac{1+\delta}{2}+(\frac{n-\mu}{2d_1}-\frac{n-\mu}{2d_6})]}\big\|f(\cdot,s)\big\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}\\ &= C(t-s)^{-(\frac{1+\delta}{2}+\frac{n-\mu}{2d_1})+\kappa(\frac{n-1-\mu}{2d_3}+\frac{\delta}{2})+(1-\kappa)(\frac{n-1-\mu}{2d_4}+\frac{\delta}{2})}\big\|f(\cdot,s)\big\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}\\ &= C(t-s)^{-\frac{1}{2}-(\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2})}\big\|f(\cdot,s)\big\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}. \end{aligned} \tag{5.20}$$

Second step (*case* $d_2=\infty$). We obtain from Jessen inequality that

$$\big|h\big(x',0,t,s\big)\big|^{d_1} \leqslant C(t-s)^{-\frac{1}{2}(d_1-1)}\int_{\mathbb{R}^n}\tilde{g}\big(x'-z',z_n,t-s\big)\big|f\big(z',z_n,s\big)\big|^{d_1}\,dz_n\,dz', \tag{5.21}$$

because $\int_{\mathbb{R}^n}\tilde{g}(x'-z',z_n,t-s)\,dz_n\,dz' = C(t-s)^{-1/2}$. Let $x_0\in\mathbb{R}^n_+$ and $[\Omega_r(x_0)]'$ be the projection of $\Omega_r(x_0)$ onto $\partial\mathbb{R}^n_+$. Consider

$$\rho\big(r,x',s\big)=\int_{|z|<r}\big|f\big(x'-z',z_n,s\big)\big|^{d_1}\,dz'\,dz_n$$

and note that

$$\rho\big(r,x',s\big)\leqslant r^{\mu}\big\|f(\cdot,s)\big\|^{d_1}_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)},\quad \text{for all } r,x',s. \tag{5.22}$$

An integration by parts in (5.21) yields

$$\begin{aligned}\big|h\big(x',0,t,s\big)\big|^{d_1} &\leqslant C(t-s)^{-\frac{1}{2}(d_1-1)}\int_{\mathbb{R}^n}\tilde{g}\big(x'-z',z_n,t-s\big)\big|f\big(z',z_n,s\big)\big|^{d_1}\,dz_n\,dz'\\ &= C(t-s)^{-\frac{1}{2}(d_1-1)}\int_0^\infty \tilde{g}(r,t-s)\,d\big[\rho\big(r,x',s\big)\big]\\ &\leqslant C(t-s)^{-\frac{1}{2}(d_1-1)}\int_0^\infty\big|\partial_r\tilde{g}(r,t-s)\big|\rho\big(r,x',s\big)\,dr. \end{aligned} \tag{5.23}$$

It follows by inserting (5.22) into (5.23) that

$$\begin{aligned}\big|h\big(x',0,t,s\big)\big|^{d_1} &\leqslant C(t-s)^{-\frac{1}{2}(d_1-1)}\int_0^\infty\big|\partial_r\tilde{g}(r,t-s)\big|r^{\mu}\,dr\,\big\|f(\cdot,s)\big\|^{d_1}_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}\\ &\leqslant C(t-s)^{-\frac{1}{2}d_1-\frac{n-\mu}{2}}\int_0^\infty\big|\partial_r\tilde{g}(r,1)\big|r^{\mu}\,dr\,\big\|f(\cdot,s)\big\|^{d_1}_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}\\ &= C(t-s)^{-\frac{1}{2}d_1-\frac{n-\mu}{2}}\big\|f(\cdot,s)\big\|^{d_1}_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}, \end{aligned}$$

which, after taking supremum over $x' \in \mathbb{R}^{n-1}$, implies

$$\left\| h\left(x', 0, t, s\right)\right\|_{L^\infty(\mathbb{R}^{n-1}, dx')} \leqslant C(t-s)^{-\frac{1}{2}-\frac{n-\mu}{2d_1}} \left\| f(\cdot, s)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}. \tag{5.24}$$

Third step. Integrating (with respect to s) either (5.20) when $d_2 < \infty$ or (5.24) when $d_2 = \infty$, we get

$$\begin{aligned}
\left\| \mathcal{F}_1[f](\cdot, t)\right\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})} &\leqslant C \int_0^t \left\| h\left(x', 0, t, s\right)\right\|_{\mathcal{M}_{d_2,\mu}(\mathbb{R}^{n-1})} ds \\
&\leqslant C \int_0^t (t-s)^{-(\frac{1}{2}+\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2})} \left\| f(\cdot, s)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)} ds \qquad (5.25) \\
&\leqslant C \int_0^t (t-s)^{-(\frac{1}{2}+\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2})} s^{-2\eta} \, ds \sup_{s>0} s^{2\eta} \left\| f(\cdot, s)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)} \\
&= C t^{-(\frac{1}{2}+\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2})-2\eta+1} \sup_{t>0} t^{2\eta} \left\| f(\cdot, t)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}, \qquad (5.26)
\end{aligned}$$

which is (5.11). □

The next lemma provides estimates for the integral operator (3.10).

Lemma 5.3. *Let $\eta \geqslant 0$, $1 < q_1 < q_2 < \infty$ and $0 \leqslant \mu < n-1$ be such that $\frac{n-1-\mu}{q_1} = \frac{n-\mu}{q_2}$. For $j = 1, 2, \ldots, n$, we have*

$$\sup_{t>0} t^\eta \left\| \mathcal{P}_j[\phi](\cdot, t)\right\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^n_+)} \leqslant C \sup_{t>0} t^\eta \left\| \phi(\cdot, t)\right\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}, \tag{5.27}$$

where $C > 0$ is independent of ϕ.

Proof. Firstly observe that $\Omega_r(x_0) = \{x \in \mathbb{R}^n_+;\ |x - x_0| < r\} \subset [\Omega_r(x_0)]' \times (0, \infty)$. It follows that

$$\begin{aligned}
\left\| \mathcal{P}_j[\phi](x, t)\right\|_{L^{q_2}(\Omega_r(x_0))} &\leqslant \left\| \mathcal{P}_j[\phi](x, t)\right\|_{L^{q_2}([\Omega_r(x_0)]' \times (0,\infty))} \\
&= \left\| \left\| \mathcal{P}_j[\phi](x, t)\right\|_{L^{q_2}((0,\infty), dx_n)}\right\|_{L^{q_2}([\Omega_r(x_0)]', dx')}. \qquad (5.28)
\end{aligned}$$

From (3.10) and (3.12), we get

$$\begin{aligned}
\left\| \mathcal{P}_j[\phi](x, t)\right\|_{L^{q_2}((0,\infty), dx_n)} &\leqslant \int_{\mathbb{R}^{n-1}} \left\| P_j(x, y)\phi(y, t)\right\|_{L^{q_2}((0,\infty), dx_n)} dy \\
&= \int_{\mathbb{R}^{n-1}} \left\| P_j(x, y)\right\|_{L^{q_2}((0,\infty), dx_n)} \left|\phi(y, t)\right| dy \\
&\leqslant C \int_{\mathbb{R}^{n-1}} \frac{|\phi(y, t)|}{|x' - y|^{n-1-\frac{1}{q_2}}} dy. \qquad (5.29)
\end{aligned}$$

Since $\frac{n-1-\mu}{q_1} - \frac{n-1-\mu}{q_2} = \frac{1}{q_2}$, we can use Lemma 2.2 to estimate

$$r^{-\frac{\mu}{q_2}} \left\| \int_{\mathbb{R}^{n-1}} \frac{|\phi(y,t)|}{|x'-y|^{n-1-\frac{1}{q_2}}} \, dy \right\|_{L^{q_2}([\Omega_r(x_0)]',dx')} \leqslant \left\| \int_{\mathbb{R}^{n-1}} \frac{|\phi(y,t)|}{|x'-y|^{n-1-\frac{1}{q_2}}} \, dy \right\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^{n-1})} \leqslant C \left\| \phi(\cdot,t) \right\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})}. \tag{5.30}$$

Therefore

$$\begin{aligned}
\sup_{t>0} t^{\eta} \left\| \mathcal{P}_j[\phi](\cdot,t) \right\|_{\mathcal{M}_{q_2,\mu}(\mathbb{R}^n_+)} &= \sup_{t>0} t^{\eta} \Big[\sup_{x_0 \in \mathbb{R}^n_+,\, r>0} r^{-\frac{\mu}{q_2}} \left\| \mathcal{P}_j[\phi](\cdot,t) \right\|_{L^{q_2}(\Omega_r(x_0))} \Big] \\
&\leqslant \sup_{t>0} t^{\eta} \Big[\sup_{x_0 \in \mathbb{R}^n_+,\, r>0} r^{-\frac{\mu}{q_2}} \left\| \left\| \mathcal{P}_j[\phi](\cdot,t) \right\|_{L^{q_2}((0,\infty),dx_n)} \right\|_{L^{q_2}([\Omega_r(x_0)]',dx')} \Big] \\
&\leqslant C \sup_{t>0} t^{\eta} \left\| \phi(\cdot,t) \right\|_{\mathcal{M}_{q_1,\mu}(\mathbb{R}^{n-1})},
\end{aligned} \tag{5.31}$$

because of the estimates (5.28), (5.29) and (5.30). □

5.2. *Estimates for the operators $\mathcal{N}$ and $\mathcal{L}$*

The next lemmas provide estimates for the linear and bilinear term of (3.17) given in (3.18):

$$\mathcal{L}[u_0] = G(t)\bar{u}_0 - \mathcal{K}\big[\big(G(t)\bar{u}_0\big)\big|_0 \big] \quad \text{and} \quad \mathcal{N}(u,u) = B(\bar{u},\bar{u}) + \mathcal{K}\big[B(\bar{u},\bar{u})|_0 \big]. \tag{5.32}$$

Lemma 5.4. *Let $1 < p < \{\frac{q(p-1)}{p}, r\} < q < \infty$, $0 \leqslant \mu < n-1$ and $p = n-\mu$. There exist $C_1, C_2 > 0$ such that*

$$\left\| \mathcal{K}[a] \right\|_{H_q} \leqslant C_1 \|a\|_{\widetilde{H}_r}, \tag{5.33}$$

$$\left\| \mathcal{L}(u_0) \right\|_{H_q} \leqslant C_2 \|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}, \tag{5.34}$$

for all $a \in \widetilde{H}_r$ and $u_0 \in \mathcal{M}_{p,\mu}(\mathbb{R}^n_+)$.

Proof. An application of (5.2) with $(\eta, q_1, q_2) = (\alpha, r, q)$ yields

$$\begin{aligned}
\sup_{t>0} t^{\alpha} \left\| \mathcal{K}_{i,j}[a_j](\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} &\leqslant C \sup_{t>0} t^{\alpha - (\frac{n-1-\mu}{2r} - \frac{n-\mu}{2q})} \left\| a(\cdot,t) \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} \\
&= C \sup_{t>0} t^{\frac{1}{2} - \frac{n-1-\mu}{2r}} \left\| a(\cdot,t) \right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} \\
&\leqslant C \|a\|_{\widetilde{H}_r},
\end{aligned} \tag{5.35}$$

for every $i, j = 1, 2, \ldots, n$. In order to treat the operator $\mathcal{P}$, we use (5.27) with $(\eta, q_1, q_2) = (\alpha, q\frac{p-1}{p}, q)$ to obtain

$$\sup_{t>0} t^{\alpha} \left\| \mathcal{P}_j[a_n](\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \leqslant C \sup_{t>0} t^{\alpha} \left\| a_n(\cdot,t) \right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} \leqslant C \|a\|_{\widetilde{H}_r}, \tag{5.36}$$

for every $j = 1, 2, \ldots, n$. In view of (3.8), the estimates (5.35) and (5.36) imply (5.33).

Now we deal with (5.34). For that matter, we use (2.10) with $(k, q_1, q_2) = (0, p, q)$ to obtain

$$\begin{aligned}
\left\|G(t)\bar{u}_0\right\|_{H_q} &= \sup_{t>0} t^\alpha \left\|G(t)\bar{u}_0\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&\leqslant \sup_{t>0} t^\alpha \left\|G(t)\bar{u}_0\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)} \\
&\leqslant C \sup_{t>0} t^\alpha t^{-\alpha} \|\bar{u}_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} \\
&\leqslant C \|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)},
\end{aligned} \tag{5.37}$$

because of (3.3). Moreover, it follows from (5.12) with $\varphi = \bar{u}_0$, $(d_1, d_2) = (p, \frac{q(p-1)}{p})$ and $(d_1, d_2) = (p, r)$ that

$$\sup_{t>0} t^\alpha \left\|\left(G(t)\bar{u}_0\right)\big|_0\right\|_{\mathcal{M}_{\frac{q(p-1)}{p},\mu}(\mathbb{R}^{n-1})} \leqslant C \sup_{t>0} t^\alpha t^{-(\frac{1}{2}-\frac{n-\mu}{2q})} \|\bar{u}_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} \leqslant C \|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}$$

and

$$\sup_{t>0} t^\beta \left\|\left(G(t)\bar{u}_0\right)\big|_0\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} \leqslant C \sup_{t>0} t^\beta t^{-(\frac{1}{2}-\frac{n-\mu-1}{2r})} \|\bar{u}_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n)} \leqslant C \|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}.$$

These two last estimates together with (5.33) imply

$$\left\|\mathcal{K}\left[\left(G(t)\bar{u}_0\right)\big|_0\right]\right\|_{H_q} \leqslant C \left\|\left(G(t)\bar{u}_0\right)\big|_0\right\|_{\widetilde{H}_r} \leqslant C \|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}. \tag{5.38}$$

We finish the proof by observing that (5.34) follows at once from (5.37) and (5.38). □

Lemma 5.5. *Let* $2 < p < \frac{q(p-1)}{p} < q < \infty$, $0 \leqslant \mu < n-2$ *and* $p = n - \mu$. *There exists a constant* $K > 0$ *such that*

$$\left\|\mathcal{N}(u, v)\right\|_{H_q} \leqslant K \|u\|_{H_q} \|v\|_{H_q}, \tag{5.39}$$

for all $u, v \in H_q$.

Proof. Recall that

$$\mathcal{N}(u, v) = B(\bar{u}, \bar{v}) + \mathcal{K}\left[B(\bar{u}, \bar{v})|_0\right]. \tag{5.40}$$

First step. Here we deal with the parcel $B(\bar{u}, \bar{v})$ of $\mathcal{N}(u, v)$. It follows from the semigroup estimate (2.10) and Hölder inequality (2.4) that

$$\begin{aligned}
\left\|B(\bar{u}, \bar{v})(t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} &\leqslant \int_0^t \left\|\nabla \cdot G(t-s)(\bar{u} \otimes \bar{v})(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} ds \\
&\leqslant C \int_0^t (t-s)^{-\frac{1}{2}(\frac{n-\mu}{q/2}-\frac{n-\mu}{q})-\frac{1}{2}} \left\|(u \otimes v)(s)\right\|_{\mathcal{M}_{\frac{q}{2},\mu}(\mathbb{R}^n_+)} ds
\end{aligned}$$

$$\leqslant C\int_0^t (t-s)^{\alpha-1}\left\|u(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\left\|v(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\,ds \tag{5.41}$$

$$\leqslant C\int_0^t (t-s)^{\alpha-1}s^{-2\alpha}\,ds\sup_{t>0}t^{\alpha}\left\|u(t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\sup_{t>0}t^{\alpha}\left\|v(t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}$$

$$\leqslant Ct^{-\alpha}\|u\|_{H_q}\|v\|_{H_q}, \tag{5.42}$$

because $\int_0^t (t-s)^{\alpha-1}s^{-2\alpha}\,ds = t^{-\alpha}[\beta(1-2\alpha,\alpha)]$ where $\beta(\cdot,\cdot)$ is the Beta function.

Second step. Now we turn to the boundary term $\mathcal{K}[B(\bar{u},\bar{v})|_0]$. For that matter, we use (5.11) with $f=\bar{u}\otimes\bar{v}$ and $(\eta,d_1,d_2)=(\alpha,\frac{q}{2},q\frac{p-1}{p})$ to obtain

$$\left\|\left(\left[B(\bar{u},\bar{v})\right]_n\right)\big|_0(\cdot,t)\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}\leqslant$$

$$\leqslant Ct^{-(\frac{n-\mu}{2d_1}-\frac{n-1-\mu}{2d_2}+2\alpha-\frac{1}{2})}\sup_{t>0}t^{2\alpha}\left\|[\bar{u}\otimes\bar{v}](\cdot,t)\right\|_{\mathcal{M}_{d_1,\mu}(\mathbb{R}^n)}$$

$$\leqslant Ct^{-(\frac{n-\mu}{q}-\frac{n-\mu}{2q}+2\alpha-\frac{1}{2})}\sup_{t>0}t^{\alpha}\left\|\bar{u}(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\sup_{t>0}t^{\alpha}\left\|\bar{v}(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}$$

$$\leqslant Ct^{-\alpha}\sup_{t>0}t^{\alpha}\left\|u(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\sup_{t>0}t^{\alpha}\left\|v(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}, \tag{5.43}$$

where above we have used that $\frac{1}{d_1}=\frac{1}{q}+\frac{1}{q}$ and Hölder inequality (2.4). Moreover, letting $\frac{q}{2}<\frac{q(p-1)}{p}<r<q$ and taking $(\eta,d_1,d_2)=(\alpha,\frac{q}{2},r)$ and $f=\bar{u}\otimes\bar{v}$ in (5.11), it follows that

$$\left\|\left[B(\bar{u},\bar{v})|_0\right](\cdot,t)\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\leqslant$$

$$\leqslant Ct^{-(\frac{n-\mu}{q}-\frac{n-1-\mu}{2r}+2\alpha-\frac{1}{2})}\sup_{t>0}t^{2\alpha}\left\|[\bar{u}\otimes\bar{v}](\cdot,t)\right\|_{\mathcal{M}_{\frac{q}{2},\mu}(\mathbb{R}^n)}$$

$$\leqslant Ct^{-(1-2\alpha-\frac{n-1-\mu}{2r}+2\alpha-\frac{1}{2})}\sup_{t>0}t^{\alpha}\left\|\bar{u}(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\sup_{t>0}t^{\alpha}\left\|\bar{v}(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}$$

$$\leqslant Ct^{-\beta}\sup_{t>0}t^{\alpha}\left\|u(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\sup_{t>0}t^{\alpha}\left\|v(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}. \tag{5.44}$$

The estimates (5.43) and (5.44) together are equivalent to

$$\left\|B(\bar{u},\bar{v})|_0\right\|_{\widetilde{H}_r}\leqslant C\|u\|_{H_q}\|v\|_{H_q}.$$

From (5.33) we get

$$\left\|\mathcal{K}\left[B(\bar{u},\bar{v})|_0\right]\right\|_{H_q}\leqslant C\left\|B(\bar{u},\bar{v})|_0\right\|_{\widetilde{H}_r}\leqslant C\|u\|_{H_q}\|v\|_{H_q}. \tag{5.45}$$

In view of (5.40), the estimate (5.39) follows from (5.42) and (5.45). $\square$

5.3. Proof of Theorem 4.1

Part (i) – Global existence. From Lemma 5.4, we have that

$$\begin{aligned}\left\|\mathcal{K}[a]+\mathcal{L}[u_0]\right\|_{H_q} &\leqslant C_1\|a\|_{\widetilde{H}_r}+C_2\|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}\\ &\leqslant (C_1+C_2)\delta=\varepsilon,\end{aligned} \tag{5.46}$$

provided that

$$\|u_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)},\ \|a\|_{\widetilde{H}_r}\leqslant\delta=\frac{\varepsilon}{C_1+C_2}.$$

Consider the ball $D_{2\varepsilon}=\{u\in H_q;\ \|u\|_{H_q}\leqslant 2\varepsilon\}$ endowed with the complete metric $\mathcal{Z}(\cdot,\cdot)$ defined by $\mathcal{Z}(u,v)=\|u-v\|_{H_q}$. For some $\varepsilon>0$, we wish to show that the map

$$\Phi(u):=\mathcal{N}(u,u)+\mathcal{K}[a]+\mathcal{L}[u_0] \tag{5.47}$$

is a contraction on $(D_{2\varepsilon},\mathcal{Z})$. Using the bilinearity of $\mathcal{N}$ and Lemma 5.5, we obtain

$$\begin{aligned}\left\|\Phi(u)-\Phi(v)\right\|_{H_q} &= \left\|\mathcal{N}(u,u)-\mathcal{N}(v,v)\right\|_{H_q}\\ &\leqslant \left\|\mathcal{N}(u-v,u)\right\|_{H_q}+\left\|\mathcal{N}(v,u-v)\right\|_{H_q}\\ &\leqslant K\|u-v\|_{H_q}\left(\|u\|_{H_q}+\|v\|_{H_q}\right)\\ &\leqslant 4\varepsilon K\|u-v\|_{H_q},\end{aligned} \tag{5.48}$$

for all $u,v\in D_{2\varepsilon}$. Moreover, it follows from (5.46) and Lemma 5.5

$$\begin{aligned}\left\|\Phi(u)\right\|_{H_q} &= \left\|\mathcal{K}[a]+\mathcal{L}(u_0)\right\|_{H_q}+\left\|\mathcal{N}(u,u)\right\|_{H_q}\\ &\leqslant \varepsilon+K\|u\|^2_{H_q}\leqslant\varepsilon+4K\varepsilon^2\\ &< 2\varepsilon,\end{aligned} \tag{5.49}$$

provided that $u\in D_{2\varepsilon}$ and $4K\varepsilon<1$. In view of (5.48)–(5.49), the map $\Phi: D_{2\varepsilon}\to D_{2\varepsilon}$ is a contraction and has a fixed point in $D_{2\varepsilon}$, which is the unique solution u for the integral equation (3.6) satisfying $\|u\|_{H_q}\leqslant 2\varepsilon$.

Next, let $u,\tilde{u}\in D_{2\varepsilon}$ satisfy (3.6) with respective initial–boundary data (u_0,a), $(\tilde{u}_0,\tilde{a})$. Because $4K\varepsilon<1$, the Lipschitz continuity of the data-solution map follows at once from

$$\begin{aligned}\|u-v\|_{H_q} &\leqslant \left\|\mathcal{K}[a-\tilde{a}]\right\|_{H_q}+\left\|\mathcal{L}[u_0-\tilde{u}_0]\right\|_{H_q}+\left\|\mathcal{N}(u,u)-\mathcal{N}(\tilde{u},\tilde{u})\right\|_{H_q}\\ &\leqslant C_1\|u_0-\tilde{u}_0\|_{\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)}+C_2\|a-\tilde{a}\|_{\widetilde{H}_r}+4K\varepsilon\|u-\tilde{u}\|_{H_q}.\end{aligned}$$

Part (ii) – Self-similarity. First we recall that homogeneous functions of degree -1 belong to the Morrey space $\mathcal{M}_{p,\mu}(\mathbb{R}^n_+)$ when $p=n-\mu$ (see Lemma 2.1). Because Theorem 4.1 (i) has been proved by means of a fixed point argument, the solution u is the limit in H_q of the following Picard interaction:

$$u_1(x,t)=\mathcal{K}[a]+\mathcal{L}[u_0]\quad\text{and}\quad u_{k+1}=u_1+\mathcal{N}(u_k,u_k) \tag{5.50}$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{N}$ are given in (3.8) and (3.18). Note that the extension operator (3.2) preserves homogeneity and scaling. Then, using that u_0 is homogeneous of degree -1 and $a(x,t)=\lambda a(\lambda x,\lambda t^2)$, one can show that

$$u_1(t,x)=\lambda u_1\big(\lambda x,\lambda^2 t\big), \tag{5.51}$$

that is, $u_1(t,x)$ is invariant by (4.1). Moreover, it is not difficult to prove that $\mathcal{N}(u,u)$ is invariant by (4.1) whenever u is also. The latter fact, property (5.51) and an induction argument yield

$$u_k(x,t)=\lambda u_k\big(\lambda x,\lambda^2 t\big) \quad \text{for all } k\in\mathbb{N}. \tag{5.52}$$

Since the norm $\|\cdot\|_{H_q}$ is invariant by (4.1) and $u_k\to u$ in H_q, it follows from (5.52) that u is self-similar. $\square$

5.4. Proof of Theorem 4.2

Subtracting the integral equations satisfied by $u(x,t)$ and $v(x,t)$, and afterwards taking the norm $t^\alpha\|\cdot\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}$, we obtain

$$\begin{aligned}
t^\alpha\big\|u(\cdot,t)-v(\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \leqslant{}& t^\alpha\big\|\mathcal{L}[u_0-v_0](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+t^\alpha\big\|\mathcal{K}[a-b](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\\
&+t^\alpha\big\|B(\bar u,\bar u)(\cdot,t)-B(\bar v,\bar v)(\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\\
&+t^\alpha\big\|\mathcal{K}\big[B(\bar u,\bar u)|_0\big](\cdot,t)-\mathcal{K}\big[B(\bar v,\bar v)|_0\big](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\\
:={}& I_0(t)+I_1(t)+I_2(t)+I_3(t).
\end{aligned} \tag{5.53}$$

Working as in the proofs of (5.7) for $\mathcal{K}_{i,j}$ and (5.31) for $\mathcal{P}_j$ with $(\eta,q_1,q_2)=(\alpha,r,q)$ and $(\eta,q_1,q_2)=(\alpha,q\frac{p-1}{p},q)$, respectively, one can obtain

$$\begin{aligned}
I_1(t)&=t^\alpha\big\|\mathcal{K}[a-b](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\\
&\leqslant Ct^\alpha\left(\sum_{j=1}^n\big\|\mathcal{P}_j[a_n-b_n](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+\sum_{i,j=1}^n\big\|\mathcal{K}_{i,j}[a-b](\cdot,t)\big\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\right)\\
&\leqslant C\Bigg(t^\alpha\big\|a_n(\cdot,t)-b_n(\cdot,t)\big\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}\\
&\qquad+t^\alpha\int_0^t(t-s)^{-\frac12-\frac{\theta}{2}(n-\frac1q)}\big\|a(\cdot,s)-b(\cdot,s)\big\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\,ds\Bigg)\\
&=C\Bigg(t^\alpha\big\|a_n(\cdot,t)-b_n(\cdot,t)\big\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}\\
&\qquad+\int_0^1(1-s)^{-\frac12-\frac{\theta}{2}(n-\frac1q)}s^{-\beta}(ts)^\beta\big\|a(\cdot,ts)-b(\cdot,ts)\big\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\,ds\Bigg),
\end{aligned} \tag{5.54}$$

where, in the last integral, we have used the change of variable $s\to st$. Moreover,

$$\begin{aligned}
I_0(t) &\leqslant t^{\alpha}\left\|G(t)[\bar{u}_0-\bar{v}_0](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+t^{\alpha}\left\|\mathcal{K}\left[G(t)(\bar{u}_0-\bar{v}_0)|_0\right](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&\leqslant t^{\alpha}\left\|G(t)[\bar{u}_0-\bar{v}_0](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+Ct^{\alpha}\left\|\left[G(t)(\bar{u}_0-\bar{v}_0)\right]_n\Big|_0\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} \\
&\quad + C\int_0^1(1-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}s^{-\beta}(ts)^{\beta}\left\|G(ts)(\bar{u}_0-\bar{v}_0)|_0\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\,ds, \qquad (5.55)
\end{aligned}$$

and

$$\begin{aligned}
I_2(t) &\leqslant t^{\alpha}\left\|B(\bar{u}-\bar{v},\bar{u})(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+t^{\alpha}\left\|B(\bar{v},\bar{u}-\bar{v})(\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&\leqslant Ct^{\alpha}\int_0^t(t-s)^{-\frac{n-\mu}{2q}-\frac{1}{2}}s^{-2\alpha}\Big[s^{\alpha}\left\|(\bar{u}-\bar{v})(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}s^{\alpha}\left\|\bar{u}(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)} \\
&\quad + s^{\alpha}\left\|\bar{v}(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}s^{\alpha}\left\|(\bar{u}-\bar{v})(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\Big]\,ds \\
&\leqslant 4\varepsilon C\int_0^1(1-s)^{-\frac{n-\mu}{2q}-\frac{1}{2}}s^{-2\alpha}(ts)^{\alpha}\left\|(u-v)(ts)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\,ds, \qquad (5.56)
\end{aligned}$$

because $\|v\|_{H_q}, \|u\|_{H_q} \leqslant 2\varepsilon$. For I_3, we have

$$\begin{aligned}
I_3(t) &\leqslant t^{\alpha}\left\|\mathcal{K}\left[B(\bar{u}-\bar{v},\bar{u})|_0\right](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}+t^{\alpha}\left\|\mathcal{K}\left[B(\bar{v},\bar{u}-\bar{v})|_0\right](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&\leqslant Ct^{\alpha}\left\|\left[B(\bar{u}-\bar{v},\bar{u})\right]_n\Big|_0(\cdot,t)\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} \\
&\quad + Ct^{\alpha}\int_0^t(t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}\left\|\left[B(\bar{u}-\bar{v},\bar{u})|_0\right](\cdot,s)\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\,ds \\
&\quad + Ct^{\alpha}\left\|\left[B(\bar{v},\bar{u}-\bar{v})\right]_n\Big|_0(\cdot,t)\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})} \\
&\quad + Ct^{\alpha}\int_0^t(t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}\left\|\left[B(\bar{v},\bar{u}-\bar{v})|_0\right](\cdot,s)\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}\,ds. \qquad (5.57)
\end{aligned}$$

Using (5.25) with $(d_1,d_2)=(\frac{q}{2},q\frac{p-1}{p})$ and $(d_1,d_2)=(\frac{q}{2},r)$, the right hand side of (5.57) can be bounded by

$$\begin{aligned}
&Ct^{\alpha}\int_0^t(t-s)^{-\frac{n-\mu}{2q}-\frac{1}{2}}\left\|\bar{u}(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\left\|(\bar{u}-\bar{v})(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\,ds \\
&\quad + Ct^{\alpha}\int_0^t(t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}\Pi_1(s)\,ds +
\end{aligned}$$

$$+ Ct^{\alpha}\int_0^t (t-s)^{-\frac{n-\mu}{2q}-\frac{1}{2}}\left\|\bar{v}(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\left\|(\bar{u}-\bar{v})(s)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\,ds$$

$$+ Ct^{\alpha}\int_0^t (t-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}\Pi_2(s)\,ds$$

$$\leqslant 4\varepsilon C\int_0^1 (1-s)^{\alpha-1}s^{-2\alpha}(ts)^{\alpha}\left\|(u-v)(ts)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\,ds$$

$$+ C\int_0^1 (1-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})}s^{-\beta}(ts)^{\beta}\big(\Pi_1(ts)+\Pi_2(ts)\big)\,ds, \tag{5.58}$$

where

$$(ts)^{\beta}\Pi_1(ts) = \int_0^{ts}(ts-\tau)^{2\alpha-\beta-1}\left\|\bar{u}(\tau)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\left\|(\bar{u}-\bar{v})(\tau)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n)}\,d\tau$$

$$\leqslant 2\varepsilon\int_0^1 (1-\tau)^{2\alpha-\beta-1}\tau^{-2\alpha}(ts\tau)^{\alpha}\left\|(u-v)(ts\tau)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\,d\tau \tag{5.59}$$

and, similarly,

$$(ts)^{\beta}\Pi_2(ts) \leqslant 2\varepsilon\int_0^1 (1-\tau)^{2\alpha-\beta-1}\tau^{-2\alpha}(ts\tau)^{\alpha}\left\|(u-v)(ts\tau)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}\,d\tau. \tag{5.60}$$

Let us set $\Gamma := \limsup_{t\to\infty} t^{\alpha}\|u(\cdot,t)-v(\cdot,t)\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}$. Taking $\limsup_{t\to\infty}$ in (5.54)–(5.60), we obtain the following inequalities:

$$\limsup_{t\to\infty}\big[I_0(t)\big] \leqslant \limsup_{t\to\infty} t^{\alpha}\left\|G(t)[\bar{u}_0-\bar{v}_0](\cdot,t)\right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)}$$

$$+ C\limsup_{t\to\infty} t^{\alpha}\left\|\big[G(t)(\bar{u}_0-\bar{v}_0)\big]_n\big|_0\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}$$

$$+ C\int_0^1 (1-s)^{-1/2-\theta(\frac{n}{2}-\frac{1}{2q})}s^{-\beta}\,ds\limsup_{t\to\infty} t^{\beta}\left\|G(t)(\bar{u}_0-\bar{v}_0)|_0\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})}$$

$$= 0+0+0 = 0, \tag{5.61}$$

$$\limsup_{t\to\infty}\big[I_1(t)+I_2(t)\big] \leqslant C\limsup_{t\to\infty} t^{\alpha}\left\|a_n(\cdot,t)-b_n(\cdot,t)\right\|_{\mathcal{M}_{q\frac{p-1}{p},\mu}(\mathbb{R}^{n-1})}$$

$$+ C\left(\int_0^1 (1-s)^{-\frac{1}{2}-\theta(\frac{n}{2}-\frac{1}{2q})}s^{-\beta}\,ds\right)\limsup_{t\to\infty} t^{\beta}\left\|a(\cdot,t)-b(\cdot,t)\right\|_{\mathcal{M}_{r,\mu}(\mathbb{R}^{n-1})} +$$

$$
\begin{aligned}
&+ 4\varepsilon C \int_0^1 (1-s)^{-\frac{n-\mu}{2q}-\frac{1}{2}} s^{-2\alpha}\, ds \limsup_{t\to\infty} t^{\alpha} \left\| (u-v)(\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&= 0 + 0 + 4\varepsilon K_1 \Gamma = 4\varepsilon K_1 \Gamma,
\end{aligned} \tag{5.62}
$$

and

$$
\begin{aligned}
\limsup_{t\to\infty} \bigl[I_3(t) \bigr] \leqslant{}& 4\varepsilon C \int_0^1 (1-s)^{\alpha-1} s^{-2\alpha}\, ds \limsup_{t\to\infty} t^{\alpha} \left\| (u-v)(\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
&+ C \int_0^1 (1-s)^{-\frac{1}{2}-\frac{\theta}{2}(n-\frac{1}{q})} s^{-\beta}\, ds \left[4\varepsilon \int_0^1 (1-\tau)^{2\alpha-\beta-1} \tau^{-2\alpha}\, d\tau \right] \\
&\times \limsup_{t\to\infty} t^{\alpha} \left\| (u-v)(\cdot,t) \right\|_{\mathcal{M}_{q,\mu}(\mathbb{R}^n_+)} \\
={}& 4\varepsilon (K_2 + K_3) \Gamma.
\end{aligned} \tag{5.63}
$$

Now, taking $\limsup_{t\to\infty}$ in (5.53), and using (5.61)–(5.63), it follows that

$$
\begin{aligned}
\Gamma &\leqslant \limsup_{t\to\infty} \bigl[I_0(t) + I_1(t) + I_2(t) + I_3(t) \bigr] \\
&\leqslant 0 + 4\varepsilon K_1 \Gamma + 4\varepsilon (K_2 + K_3) \Gamma \leqslant (4\varepsilon K) \Gamma,
\end{aligned}
$$

which implies $\Gamma = 0$ (because $4K\varepsilon < 1$), and then we obtain (4.9). □

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