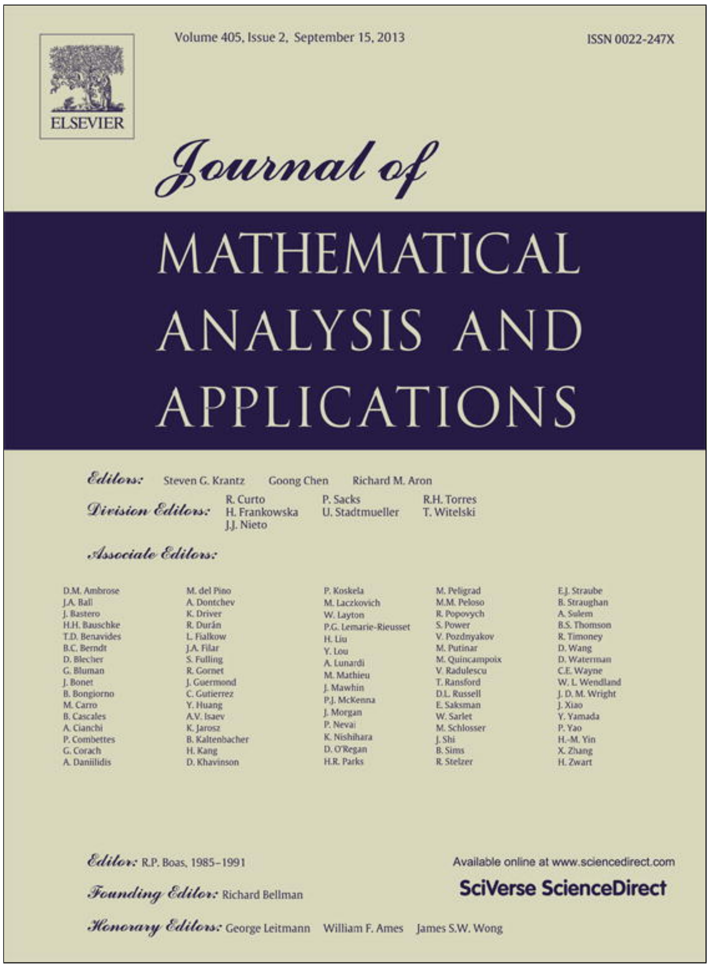
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On the local solvability in Morrey spaces of the Navier–Stokes equations in a rotating frame

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ABSTRACT

We prove local-in-time (non-uniform) solvability for the rotating Navier–Stokes equations in Morrey spaces $\mathcal{M}^{\sigma}_{p,\mu}(\mathbb{R}^3)$. These spaces contain singular and nondecaying functions which are of interest in statistical turbulence. We give an algebraic relation between the size of existence time and angular velocity Ω. The evolution of velocity u is analyzed in suitable Kato–Fujita spaces based on Morrey spaces. We show the asymptotic behavior $u_{\Omega} \to w$ in $L^{\infty}(0, T; \mathcal{M}^{\sigma}_{p,\mu}(\mathbb{R}^3))$ as $\Omega \to 0$, where w is the solution for the Navier–Stokes equations with the same data u_0. Particularly, for $\mu = 3 - p$, the solution is approximately self-similar for small $|\Omega|$, when u_0 is homogeneous of degree -1.

1. Introduction

In this paper we consider the initial value problem (IVP) for the incompressible rotating Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \Omega \mathbb{J}u + \nabla p = 0, \quad x \in \mathbb{R}^3,\ t > 0 \tag{1.1}$$

$$\nabla \cdot u = 0, \quad x \in \mathbb{R}^3,\ t \geq 0 \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3, \tag{1.3}$$

where u is the velocity field of fluid and the scalar function p denotes the pressure at the point $x \in \mathbb{R}^3$ and $t > 0$. The term $\Omega \in \mathbb{R}$ is the so-called Coriolis parameter which corresponds twice the speed of rotation around the vertical axis $e_3 = (0, 0, 1)$. The operator $\mathbb{J}$ is the skew-symmetric 3×3 matrix

$$\mathbb{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.4}$$

and the term $\Omega \mathbb{J} u = \Omega e_3 \times u$ is called Coriolis force. After applying the Leray projector $\mathbb{P}$, Duhamel's principle can be used to formally convert (1.1)–(1.3) to the integral equation

$$u(t) = \mathcal{G}(t)u_0 + \int_0^t \nabla \cdot \mathcal{G}(t-s)\mathbb{P}(u \otimes u)(s)ds, \tag{1.5}$$

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where $\{\mathcal{G}(t)\}_{t\geq 0}$ stands for the semigroup associated to linear part of (1.1)–(1.2) (see Section 2.3 for details). The operator $\mathcal{G}(t)$ is the composition of the heat semigroup $\exp(t\Delta)$ with $\exp(-t\Omega S)$ where $S = \mathbb{P}\mathbb{J}\mathbb{P}$ is the Coriolis operator, that is

$$\mathcal{G}(t)u_0 = \exp(t\Delta)\exp(-t\Omega S)u_0. \tag{1.6}$$

The family $\{\exp(-t\Omega S)\}_{t\geq 0}$ is also known as Poincaré–Riesz semigroup. Throughout this manuscript, vector fields u verifying (1.5) are called mild solutions for the Cauchy problem (1.1)–(1.3).

Due to their potential applications, fluids under rotational effects have recently attracted the attention of several authors, see e.g. [1,2,4,9,10,12,14,11,15,13,21,26]. For instance, the above system has applications connected to the large-scale ocean and atmosphere dynamics which can present sharp movements of rotation (see [5]). In the mathematical theory of rotating fluids, it is important to study local or global solvability by analyzing how the angular velocity Ω affects the size of existence time or initial data. Also, existence results for (1.1)–(1.3) with singular and nondecaying initial data can be important in the context of homogeneous statistical turbulence (see e.g. [1,2,8,14,30]).

In this paper, we obtain results about local-in-time solvability for (1.1)–(1.3) in Morrey spaces $\mathcal{M}^{\sigma}_{p,\mu}(\mathbb{R}^3)$ with $1 < p < \infty$ and $\mu \geq 3 - p$ (see Theorem 3.1), whose elements can be strongly singular. These spaces contain functions f that do not vanish as $|x| \to \infty$ (nondecaying functions) in the sense that the set

$$\{x \in \mathbb{R}^n : |x| > L \text{ and } |f(x)| > \eta\} \tag{1.7}$$

has infinite measure, for every $L > 0$ and some fixed $\eta > 0$. The evolution of the velocity is analyzed in suitable Kato–Fujita norms based on Morrey spaces. In spite of (1.1) does not have a scaling when $\Omega \neq 0$, we can use the Navier–Stokes one $u \to u_\lambda = \lambda u(\lambda x, \lambda^2 t)$ as a suitable intrinsic scaling for (1.1) in order to define a notion of critical spaces. In Theorem 3.4, we show that the solution $u_\Omega(x,t)$ is approximately self-similar when $\mu = 3 - p$, u_0 is homogeneous of degree -1, and $|\Omega|$ is small enough. Precisely, it is proved that $u_\Omega \to w$ in $L^\infty(0,T;\mathcal{M}^{\sigma}_{p,\lambda}(\mathbb{R}^3))$ as $\Omega \to 0$, for arbitrary fixed $T > 0$, where w is the self-similar global solution of the Navier–Stokes equations ($\Omega = 0$) with initial data u_0.

From another point of view, our results provide local solvability of the Navier–Stokes equations in $\mathbb{R}^3$ (3DNS) with measure as initial vorticity. Indeed, we can consider initial vorticity belonging to $\mathcal{M}_{1,1}$ which contains measure concentrated on smooth compact curves (see [16]), because Biot–Savart law implies

$$u_0(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-z|^3}(x-z)\times\omega_0(z)dz \in \mathcal{M}^{\sigma}_{2,1} \quad \text{for } \omega_0 = \nabla\times u_0 \in \mathcal{M}_{1,1}(\mathbb{R}^3).$$

Here we denote spaces of vector-valued and scalar functions in the same way.

Let us review some works about local and global solvability for (1.1)–(1.3). Local-in-time existence of solutions in the Besov space $\dot{\mathcal{B}}^0_{\infty,1}$ was proved in [18,32] with existence time T depending on Coriolis parameter Ω (i.e. non-uniform). Indeed the paper [18] considered the Eq. (1.1) with a additional drift term $Mx\cdot\nabla u$ with M a real matrix. In the paper [11], authors showed local non-uniform solvability in L^∞_{av}, which is a subspace of L^∞ having vertical averaging property. Precisely,

$$L^\infty_{av}(\mathbb{R}^3) = \left\{u \in L^\infty(\mathbb{R}^3) : u - \overline{u} \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^3)\right\} \quad \text{where } \overline{u} = \lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L} u(x_1,x_2,x_3)dx_3. \tag{1.8}$$

The motivation for introducing the space $L^\infty_{av}(\mathbb{R}^3)$ was that the Stokes–Coriolis semigroup is unbounded on $L^\infty(\mathbb{R}^3)$. Afterwards the authors of [12] introduced the space FM_0 and showed a result of local-in-time solvability, uniformly on Ω, where

$$FM_0 = \{\hat{f} : f \in \mathcal{M} \text{ has no point mass at } x = 0\},$$

and $\mathcal{M}$ is the space of finite Radon measures. We have the inclusions $FM_0 \subset \dot{\mathcal{B}}^0_{\infty,1} \subset BUC$. Concerning small global solvability for (1.1)–(1.3), uniformly on Ω, we have results on Sobolev space $H^{\frac{1}{2}}$ [19], $FM_\delta = \{f \in FM;\ \text{supp}(\hat{f}) \subset F_\delta\}$ [13], $(FM_0)^{-1} = \text{div}\left[(FM_0)^3\right]$ [14], $F\dot{B}^{2-\frac{3}{p}}_{p,\infty}$ with $p > 3$ [26] and $\dot{\mathcal{B}}^{1/2,s_p}_{2,p}$ [6] with $s_p = -1+\frac{3}{p}$ and $3 < p < \infty$. Here F_δ is a sum-closed frequency set, $FM = (\mathcal{M})^\wedge = \{\hat{f} : f \in \mathcal{M}\}$, $F\dot{B}^s_{p,\infty} = (\dot{B}^s_{p,\infty})^\vee$ denotes Fourier Besov spaces, and $\dot{\mathcal{B}}^{1/2,s_p}_{2,p}$ is a homogeneous hybrid-Besov space. The time analyticity of local solutions was studied in [15] by assuming values in the space of bounded uniformly continuous functions $BUC(\mathbb{R}^3)$ when $\Omega = 0$ and in FM_0 when $\Omega \neq 0$. Another problem related to (1.1)–(1.3) is to describe the Navier–Stokes flow past rotating obstacles, whose mathematical structure have a drift term besides the Coriolis force. In the literature for this model, the reader can find existence results for decaying data in L^p and weak-L^p (see [9,10,20,21] and references therein).

The continuous inclusions

$$W^{\frac{1}{2},2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \subset L^{3,\infty}(\mathbb{R}^3) \subset \mathcal{M}_{2,1}(\mathbb{R}^3) \tag{1.9}$$

hold true, where the first one in (1.9) follows from Sobolev embedding. On the other hand, there is no inclusion relation between $\mathcal{M}_{2,1}(\mathbb{R}^3)$ (or more generally $\mathcal{M}_{p,\mu}$) and the spaces $FM_\delta \subset FM_0 \subset (FM_0)^{-1}$, $F\dot{B}^{2-\frac{3}{p}}_{p,\infty}$ and $\dot{\mathcal{B}}^{1/2,s_p}_{2,p}$(see Remark 3.3). The same occurs between $\mathcal{M}_{p,\mu}(\mathbb{R}^3)$ and $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^3)$, $L^\infty_{av}(\mathbb{R}^3)$, for $1 \leq p < \infty$ and $0 \leq \mu < 3$. Then we are providing a new initial data class for IVP (1.1)–(1.3), which contains in particular nondecaying functions at infinity. Another types of nondecaying data have been considered in the works [12,14,11,15,13,22,32,35]. Due to the presence of Poincaré–Riesz semigroup, the analysis of rotating fluid models in a nondecaying framework requires involved arguments and, unlike L^2 and H^s-spaces, it is not possible to use energy methods.

Technically speaking, the works [12,14,13,19,26] have analyzed (1.1)–(1.3) in spaces based on Fourier transform, among other reasons, in order to obtain results uniform on Ω. We show the local-in-time solvability in a space whose norm is not characterized via Fourier transform, however our existence time depends on Ω. The Navier–Stokes equations in $\mathbb{R}^3$ have been studied in Morrey spaces by [16,24,33] where the kernel of the integral mild formulation is the heat fundamental solution which is a L^1-function. In virtue of Poincaré–Riesz semigroup $\exp(-t\Omega S)$, the kernel of the integral operator associated to (1.6) does not belong to $L^1(\mathbb{R}^3)$ since it behaves like $1/|x|^3$ as $|x| \to \infty$ (see [11, Appendix A]). In order to overcome this lack of integrability, a non-straightforward adaptation from arguments of [16,24,33] is performed here.

Finally, we recall results about existence of global solutions for large $|\Omega|$ and regardless the size of data u_0. In [1], the authors worked in Sobolev spaces of periodic functions H^{α}_{per} ($\alpha > 1/2$) with zero average and obtained solution for (1.1)–(1.3) on the time-interval $[0, \infty)$ for $|\Omega| \geq C_0$ with C_0 depending on the spatial lattice and u_0 (see also [2] for L^2-periodic data). Results for H^1-solutions and bounded cylindrical domains $U = \{x \in \mathbb{R}^3; x_3 \in [0, h], x_1^2 + x_2^2 \leq a^2\}$ can be found in [30] with C_0 depending on h/a and $\|\nabla \times u_0\|_{L^2(U)}$; there, slip boundary conditions for u on vertical plates are assumed. The author of [35] showed existence of mild solution on $[0, T]$, for any $0 < T < \infty$ and arbitrary almost periodic data, by assuming $|\Omega| \geq C_0$ where C_0 depends on T and u_0. These three last results show that fast angular velocities tend to smooth out flows of fluids. Further results for the above and other rotating fluid models can be found in [4,5].

The plan of this paper is as follows. In the next section we review some basic properties about Morrey spaces and Stokes–Coriolis semigroup. We state our results in Section 3 and prove them in Section 4. The core linear estimates for the Stokes–Coriolis semigroup on Morrey spaces are proved in Section 4.1.

2. Preliminaries

2.1. Morrey spaces

For $1 \leq p < \infty$ and $0 \leq \mu < n$, the Morrey space $\mathcal{M}_{p,\mu} = \mathcal{M}_{p,\mu}(\mathbb{R}^n)$ is the space of all measurable functions such that

$$\|f\|_{p,\mu} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\mu/p} \|f\|_{L^p(B_r(x_0))} < \infty, \tag{2.1}$$

where $B_r(x_0) \subset \mathbb{R}^n$ is the closed ball in $\mathbb{R}^n$ with center x_0 and radius r. Notice that $\mathcal{M}_{p,0} = L^p$ for $p > 1$ and $\mu = 0$. By identifying locally integrable functions with Radon measure, $\mathcal{M}_{1,\mu}(\mathbb{R}^n)$ is a set of Radon measures on $\mathbb{R}^n$, and the space $\mathcal{M}_{1,0}(\mathbb{R}^n)$ denotes the set of finite real valued Radon measure. Finally the cases $\mu = n$ and $\mu > n$ correspond to $L^\infty(\mathbb{R}^n)$ and to the null space $\{0\}$, respectively. For further details about Morrey spaces, see [24,29,31,27,28]. The space $\mathcal{M}_{p,\mu}$ endowed with the norm $\|\cdot\|_{p,\mu}$ is a Banach space and

$$\|f(lx)\|_{p,\mu} = l^{-\frac{n-\mu}{p}} \|f(x)\|_{p,\mu}, \quad \text{for } l > 0.$$

In the next lemma we recall some basic inequalities in Morrey spaces (see [24]).

Lemma 2.1. *Assume that* $1 \leq p, q, r < \infty$ *and* $0 \leq \lambda, \mu, \upsilon < n$.

(i) *(Inclusion). If* $\frac{n-\lambda}{p} = \frac{n-\mu}{q}$ *and* $p \leq q$ *then*

$$\mathcal{M}_{q,\mu} \subset \mathcal{M}_{p,\lambda}. \tag{2.2}$$

(ii) *(Hölder inequality). If* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and* $\frac{\upsilon}{r} = \frac{\lambda}{p} + \frac{\mu}{q}$ *then*

$$\|fg\|_{r,\upsilon} \leqslant \|f\|_{p,\lambda} \|g\|_{q,\mu}. \tag{2.3}$$

(iii) *(Homogeneous function). Let* $\varphi \in L^\infty(\mathbb{S}^{n-1})$, $0 < d < n$ *and* $1 \leq p < n/d$. *Then* $\varphi(x/|x|)|x|^{-d} \in \mathcal{M}_{p,n-dp}(\mathbb{R}^n)$.

2.2. Fourier multipliers

Let $N \in \mathbb{N}$ with $N > \frac{n}{2}$, and denote $\Sigma_0^1(\mathbb{R}^n)$ by the class of C^N-functions on $\mathbb{R}^n \setminus \{0\}$ satisfying the estimate

$$\sup_{\xi \neq 0} |\xi|^{|\alpha|} |\partial_\xi^\alpha \sigma(\xi)| \leq L, \tag{2.4}$$

for all multi-index α with $|\alpha| \leq N$ (see [17,27,34]). In such a case, we have that the operator

$$T_\sigma u = \mathcal{F}^{-1} \sigma(\xi) \mathcal{F} u \tag{2.5}$$

is a Fourier multiplier on L^p and there exists a constant $C > 0$ such that (see e.g. [17, p. 362])

$$\|T_\sigma u\|_{L^p(\mathbb{R}^n)} \leq C\, L\, \|u\|_{L^p(\mathbb{R}^n)}, \tag{2.6}$$

for all $u \in L^p(\mathbb{R}^n)$. In what follows, the class $OP\Sigma_0^1(\mathbb{R}^n)$ denotes the set of operators with symbol $\sigma(T_\sigma) = \sigma(\xi)$ belonging to $\Sigma_0^1(\mathbb{R}^n)$.

In particular, the next lemma extends (2.6) to Morrey spaces. Its proof can be reached by using results and arguments of [27,33,34].

Lemma 2.2. *Let $1 < p < \infty$, $0 \le \mu < n$ and $T_\sigma \in OP\Sigma_1^0(\mathbb{R}^n)$.*

(i) *There exists $C > 0$ (independent of L) such that*

$$\|T_\sigma f\|_{p,\mu} \le CL\|f\|_{p,\mu}, \tag{2.7}$$

for all $f \in \mathcal{M}_{p,\mu}(\mathbb{R}^n)$.

(ii) *If $k_\sigma(z) = (\sigma(\xi))^\vee$ satisfies $|k_\sigma(z)| \le A_1 |z|^{-n}$ and*

$$\|T_\sigma f\|_{L^p(\mathbb{R}^n)} \le A_2\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.8}$$

for all $f \in L^p(\mathbb{R}^n)$, then

$$\|T_\sigma f\|_{p,\mu} \le \widetilde{C}\|f\|_{p,\mu}, \tag{2.9}$$

for all $f \in \mathcal{M}_{p,\mu}(\mathbb{R}^n)$, where $\widetilde{C} = C(A_1 + A_2)$ with $C > 0$ depending only on p, μ, n.

Proof. From [33, p. 1420] and [34, Propositions B.1 and B.2], we have that T_σ is a convolution operator with kernel $k_\sigma(z) = (\sigma(\xi))^\vee$ and the estimate (2.7) holds (see also [27,28]). Moreover, one can use [34, Propositions B.1 and B.2] and follow the same steps of its proofs to see that the constant $\widetilde{C} > 0$ can be taken as in (2.9). In the sequel we give some details for the reader's convenience. First one splits $f \in \mathcal{M}_{p,\mu}(\mathbb{R}^n)$ as

$$f = f_0 + \sum_{j=1}^{\infty} g_j,$$

where

$$f_0 = \chi_{B_{2r}(x_0)} f, \qquad g_j = f\chi_{A_{rj}} \quad \text{and} \quad A_{rj} = \{x : 2^j r \le |x_0 - x| \le 2^{j+1} r\}.$$

Defining $k_j(x,y) = \chi_{B_r(x_0)}(x)k(x-y)\chi_{A_{rj}}(y)$ and $T_{\sigma,j}(f) = \int_{\mathbb{R}^n} k_j(x,y)f(y)dy$, one can estimate

$$\begin{aligned}
\|T_\sigma f\|_{L^p(B_r(x_0))} &\le \|T_\sigma f_0\|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^{\infty} \left\|T_\sigma g_j\right\|_{L^p(B_r(x_0))} \\
&\le A_1 \|f_0\|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^{\infty} \left\|T_{\sigma,j}(\chi_{A_{rj}} f)\right\|_{L^p(\mathbb{R}^n)} \\
&\le A_1 \|f\|_{L^p(B_{2r}(x_0))} + \sum_{j=1}^{\infty} CA_2 2^{-jn/p} \left\|\chi_{A_{rj}} f\right\|_{L^p(\mathbb{R}^n)} \\
&\le 2^{\frac{\mu}{p}} A_1 \|f\|_{p,\mu} r^{\frac{\mu}{p}} + CA_2 \sum_{j=1}^{\infty} 2^{-jn/p} \|f\|_{p,\mu} (2^j r)^{\frac{\mu}{p}} \\
&\le C\left(1 + \sum_{j=1}^{\infty} 2^{-j\left(\frac{n-\mu}{p}\right)}\right)(A_1 + A_2)\|f\|_{p,\mu} r^{\frac{\mu}{p}},
\end{aligned} \tag{2.10}$$

which yields (2.9), because the series in (2.10) is convergent. □

Remark 2.3. Let $\sigma(\xi) \in \Sigma_1^0(\mathbb{R}^n)$ be homogeneous of degree zero. We have that $|k_\sigma(z)| \le CL|z|^{-n}$ with $C > 0$ independent of z and σ (see [3, Theorem 1] and [17, Chapters 2 and 4]). Then we can obtain (2.7) directly from (2.6) and item (ii) of Lemma 2.2.

2.3. Stokes–Coriolis semigroup

Let us recall the Stokes–Coriolis semigroup $\{\mathcal{G}(t)\}_{t\ge 0}$ associated to the linear system

$$\partial_t u - \Delta u + \Omega e_3 \times u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times \{t > 0\} \tag{2.11}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3 \times \{t \ge 0\} \tag{2.12}$$

$$u(x,0) = u_0 \quad \text{in } \mathbb{R}^3. \tag{2.13}$$

Hieber and Shibata [19, Proposition 2.1] gave an explicit representation formula for $\{\mathcal{G}(t)\}_{t\geq 0}$ by handling the corresponding resolvent equation in Fourier variables. Using infinite series for the exponential Coriolis operator, the authors of [11] obtained the same formula, namely

$$\widehat{\mathcal{G}(t)u_0} = e^{-t|\xi|^2}\left[\cos\left(t\Omega\frac{\xi_3}{|\xi|}\right)\mathcal{I} + \sin\left(t\Omega\frac{\xi_3}{|\xi|}\right)\widehat{R}(\xi)\right]\widehat{u}_0, \tag{2.14}$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$ and $t \geq 0$, where $\mathcal{I}$ is the identity in $\mathbb{R}^3$. Here $\widehat{R}(\xi)$ is a 3×3 skew-symmetric matrix whose elements are symbols of Riesz operators $\mathcal{R}_j$, $j = 1, 2, 3$. Precisely,

$$\widehat{R}(\xi) = \begin{bmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{bmatrix} \quad \text{and} \quad \widehat{R}(\xi)u := -\frac{1}{|\xi|}\xi \times u.$$

The elements of the symbol matrix of Leray projector $\mathbb{P}u = (-\Delta)^{-1}\nabla\times\nabla\times u$ belong to $\Sigma_1^0(\mathbb{R}^3)$ and are given by

$$(\widehat{\mathbb{P}}(\xi))_{i,j} = \delta_{ij} - \xi_i\xi_j/|\xi|^2. \tag{2.15}$$

It follows that $\mathbb{P}$ is bounded from $\mathcal{M}_{p,\mu}$ into itself, for $1 < p < \infty$ and $0 \leq \mu < 3$. Applying $\mathbb{P}$ in (2.11), we obtain

$$\partial_t u - \Delta u + \Omega S u = 0 \quad \text{in } \mathbb{R}^3 \times \{t > 0\} \quad \text{and} \quad u(x, 0) = u_0 \quad \text{in } \mathbb{R}^3, \tag{2.16}$$

where $S = \mathbb{P}\mathbb{J} = \mathbb{P}\mathbb{J}\mathbb{P}$ is the Coriolis operator on divergence-free vector fields and $\mathbb{J}$ is as in (1.4). Therefore

$$u(x, t) = \mathcal{G}(t)u_0 := \exp(t\Delta)\exp(-t\Omega S)u_0 \tag{2.17}$$

is a solution for (2.16) on divergence-free vector spaces and we have

$$\widehat{\exp(-t\Omega S)u_0} = \left[\cos\left(t\Omega\frac{\xi_3}{|\xi|}\right)\mathcal{I} + \sin\left(t\Omega\frac{\xi_3}{|\xi|}\right)\widehat{R}(\xi)\right]\widehat{u}_0. \tag{2.18}$$

3. Main results

In what follows, for $T > 0$, $BC((0, T), X)$ denotes the space of bounded continuous functions from $(0, T)$ to the Banach space X.

We are going to employ Kato–Fujita method (see [23,25]) to the integral equation (1.5) on Morrey spaces. To do this, we perform a scaling analysis in order to find suitable Kato–Fujita spaces based on Morrey spaces. As pointed out in Introduction, for $\Omega \neq 0$ the system (1.1)–(1.2) does not have a scaling invariance property, however we can use an "intrinsic scaling" which comes from Navier–Stokes equations, namely

$$u(x, t) \to u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t). \tag{3.1}$$

In a natural way, making $t \to 0^+$, the map (3.1) induces the following scaling for initial data

$$u_0(x) \to \lambda u_0(\lambda x). \tag{3.2}$$

Let us denote $\mathcal{M}^\sigma_{p,\mu} = \{u_0 \in \mathcal{M}_{p,\mu} : \nabla\cdot u_0 = 0\}$. For $1 < p < q < \infty$, $0 \leq \mu < 3$ and $\alpha = \frac{3-\mu}{p} - \frac{3-\mu}{q}$, we consider the Kato–Fujita type space based on Morrey spaces

$$H_{q,T} = \left\{u(\cdot, t) \in BC((0, T), \mathcal{M}^\sigma_{p,\mu}) : t^{\frac{\alpha}{2}}u(\cdot, t) \in BC((0, T), \mathcal{M}_{q,\mu})\right\}, \tag{3.3}$$

which is a Banach space with norm

$$\|u\|_{H_{q,T}} = \sup_{0<t<T}\|u(\cdot, t)\|_{p,\mu} + \sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(\cdot, t)\|_{q,\mu}. \tag{3.4}$$

Above, the upper index σ stands for solenoidal fields. Notice that $H_{q,\infty}$ is critical for the scaling (3.1) only when $\mu = 3 - p$, that is

$$\|u\|_{H_{q,\infty}} = \|u_\lambda\|_{H_{q,\infty}}. \tag{3.5}$$

For $\Omega = 0$, a solution w is called self-similar when it is invariant by (3.1), that is, $w = w_\lambda := \lambda w(\lambda x, \lambda^2 t)$, for all $\lambda > 0$. Our local-in-time solvability result reads as follows.

Theorem 3.1. *Let $1 < q' < p < q < \infty$, $\Omega \in \mathbb{R}$, $0 \leq \mu < 3$ with $\mu \geq 3 - p$ and $u_0 \in \mathcal{M}^\sigma_{p,\mu}$.*

(i) *Let $\mu > 3 - p$. There exists $C > 0$ independent of Ω and u_0, $T_\Omega := T(\Omega) > 0$, and a unique local-in-time mild solution $u \in H_{q,T_\Omega}$ for the IVP (1.1)–(1.3) satisfying $\|u\|_{H_{q,T_\Omega}} \leq 2\gamma_\Omega$, where $\gamma_\Omega = C(1 + T_\Omega|\Omega|)^2\|u_0\|_{p,\mu}$. The data-map solution is locally Lipschitz continuous from $\mathcal{M}^\sigma_{p,\mu}$ to H_{q,T_Ω}.*

(ii) *Let $\mu = 3-p$ and $\gamma_\Omega = C(1+T|\Omega|)^2\|u_0\|_{p,\mu}$, where $T > 0$ is arbitrary and C is as in item (i). There exist $\delta := \delta(T, \Omega) > 0$ and a unique mild solution $u \in H_{q,T}$ satisfying $\|u\|_{H_{q,T}} \le 2\gamma_\Omega$ provided that $\|u_0\|_{p,\mu} < \delta$. For $\Omega = 0$, there is $\delta_0 > 0$, $\delta_0 \ge \delta$, such that if $\|u_0\|_{p,\mu} < \delta_0$ then we can take $T = \infty$ and u is the unique global solution verifying $\|u\|_{H_{q,\infty}} \le 2\gamma_0 = 2C\|u_0\|_{p,\mu}$. In the first conclusion of this item, we can replace δ by δ_0 provided that $T\,|\Omega|$ is small enough.*

Remark 3.2. If $T_0 > 0$ is the existence time in the above item (i), corresponding to $\Omega = 0$, we can take $T_\Omega = T_0$ for $|\Omega|$ small enough (see (4.18)). Also, it follows from item (ii) that, for $\|u_0\|_{p,\mu} < \delta_0$ with $\mu = 3-p$, there is a solution $u_\Omega(x,t) \in H_{q,T_\Omega}$, where $T_\Omega\,|\Omega|$ is small, and $T_\Omega \to \infty$ when $\Omega \to 0$.

The case $\Omega = 0$ in item (ii) recovers an existence result due to [24].

Remark 3.3. Let $\phi \in C_0^\infty$, $\phi \ge 0$, $\phi(0) = 1$, $\int_{\mathbb{R}^3}\phi dx = 1$, and $\phi(x) = 0$ for $|x| \ge 1$. Let $\{\lambda_j\} \subset \mathbb{R}^n$ satisfy $|\lambda_j| = 4^j$ and define

$$f = \sum_{j=1}^{\infty} e^{i\lambda_j\cdot x}\phi(x-\lambda_j). \tag{3.6}$$

Then f is a nondecaying function and it belongs to $\mathcal{M}_{p,\mu}(\mathbb{R}^3)$, for $1 \le p < \infty$ and $0 \le \mu < 3$, but not to FM_δ, FM_0, $(FM_0)^{-1}$, $\dot{\mathcal{B}}^{1/2,s_p}_{2,p}$ nor $F\dot{B}^{2-\frac{3}{p}}_{p,\infty}$. In [7, Remark 2.1], the function (3.6) was used as an example of f that does not belong to weak-L^p spaces and to pseudo-measure spaces $\mathcal{PM}^a$.

Theorem 3.4. *Under the hypotheses of Theorem 3.1.*

(i) *(Vanishing angular velocity limit). Let u_Ω be the solution corresponding to angular velocity Ω, and let w be the solution of the Navier–Stokes equations ($\Omega = 0$) both with the initial data u_0. Then*

$$u_\Omega \to w \quad \text{in } L^\infty(0,T;\mathcal{M}_{p,\mu}) \text{ as } \Omega \to 0, \tag{3.7}$$

where either $T > 0$ is arbitrary if $\mu = 3-p$ or $T = T_0$ if $\mu > 3-p$ (see Remark 3.2).

(ii) *(Approximate self-similarity as $\Omega \to 0$). Assume $\mu = 3-p$. Let $u_\Omega(x,t)$ be the solution with data u_0 homogeneous of degree -1 and with existence time T_Ω, where $\|u_0\|_{p,\mu} \le \delta_0$ and $T_\Omega \to \infty$ as $\Omega \to 0$. Then, for small values of $|\Omega|$, u_Ω is approximately self-similar in $L^\infty_{\mathrm{loc}}(0,\infty;\mathcal{M}_{p,\mu})$, that is, $u_\Omega(x,t)$ converges in the sense of (3.7), for any fixed $T > 0$, to the self-similar solution w of the Navier–Stokes equations.*

4. Proofs

4.1. Linear estimates

In this section we obtain estimates for the Stokes–Coriolis semigroup $\{\mathcal{G}(t)\}_{t\ge0}$ acting on Morrey spaces. For that, we start by providing estimates for the evolution operator $\exp(t\Omega\mathcal{R}_j)$ which is called Riesz semigroup (see [22]). Here $\mathcal{R}_j$ stands for the so-called j-th Riesz transform whose symbol is

$$\sigma_j(\xi) = i\xi_j/|\xi|, \quad j = 1,\ldots,n.$$

For $1 < p < \infty$ and $0 \le \mu < n$, notice that $\mathcal{R}_j$'s are continuous on $\mathcal{M}_{p,\mu}$ because $\sigma_j \in \Sigma_1^0(\mathbb{R}^n)$ (see Lemma 2.2).

The next lemma deals with the Riesz semigroup in Morrey spaces.

Lemma 4.1. *Let $1 < p < \infty$, $0 \le \mu < n$ and $\Omega \in \mathbb{R}$. For each fixed $t \ge 0$, the operator $\exp(t\Omega\mathcal{R}_j)$ is bounded from $\mathcal{M}_{p,\mu}(\mathbb{R}^n)$ into itself, for $j = 1,\ldots,n$. Moreover, there is $C > 0$ (independent of Ω) such that*

$$\|\exp(t\Omega\mathcal{R}_j)f\|_{p,\mu} \le C\,(1+t|\Omega|)^{\lfloor n/2\rfloor+1}\|f\|_{p,\mu}, \tag{4.1}$$

for all $f \in \mathcal{M}_{p,\mu}$ and $t \ge 0$, where $\lfloor\cdot\rfloor$ stands for the greatest integer function.

Proof. Because $\sigma(\mathcal{R}_j)$ satisfies (2.4), a simple computation shows that

$$|\partial_\xi^\alpha \exp(t\Omega\,\sigma(\mathcal{R}_j))| \le C(1+t|\Omega|)^{|\alpha|}|\xi|^{-|\alpha|},$$

for all $|\alpha| \le N = \lfloor n/2\rfloor + 1$ and $j = 1,\ldots,n$. It follows that

$$\sup_{\xi\in\mathbb{R}^n}\left(|\xi|^{|\alpha|}|\partial_\xi^\alpha \exp(t\Omega\,\sigma(\mathcal{R}_j))|\right) \le L = C(1+t|\Omega|)^{\lfloor n/2\rfloor+1},$$

which implies that $\exp(t\Omega\mathcal{R}_j) \in OP\,\Sigma_0^1(\mathbb{R}^n)$. Now an application of Lemma 2.2(i) gives us the desired statement. □

In the following we recall estimates on Morrey spaces for the heat semigroup $\exp(t\Delta)$ found in [24, Lemma 2.1] (see also [33]).

Lemma 4.2. *Let $1 \leq q_1 \leq q_2 < \infty, 0 \leq \mu < n, \eta_i = \frac{n-\mu}{q_i}$ $(i = 1, 2)$ and let $\beta \in (\{0\} \cup \mathbb{N})^n$ be a multi-index. Then the operator $\partial_x^\beta \exp(t\Delta)$ is bounded from $\mathcal{M}_{q_1,\mu}$ to $\mathcal{M}_{q_2,\mu}$ and there is $C > 0$ such that*

$$t^{\frac{1}{2}(\eta_1-\eta_2)+\frac{|\beta|}{2}} \|\partial_x^\beta \exp(t\Delta)f\|_{q_2,\mu} \leq C\|f\|_{q_1,\mu},$$

for all $f \in \mathcal{M}_{q_1,\mu}$ and $t > 0$.

Lemma 4.3. *Let $n = 3$, $1 < p < \infty, 0 \leq \mu < 3$ and $\Omega \in \mathbb{R}$. Then there is $C > 0$ (independent of Ω) such that*

$$\|\exp(-t\Omega S)f\|_{p,\mu} \leq C(1 + t|\Omega|)^2\|f\|_{p,\mu}, \tag{4.2}$$

for all $f \in \mathcal{M}^\sigma_{p,\mu}(\mathbb{R}^3)$.

Proof. First observe that the real and imaginary parts of $\exp(t\Omega\sigma(\mathcal{R}_3))$ are $\cos(t\Omega\xi_3/|\xi|) = \cos(t\Omega i\sigma(\mathcal{R}_3))$ and $\sin(t\Omega\xi_3/|\xi|) = -\sin(t\Omega i\sigma(\mathcal{R}_3))$, respectively, which belong to $OP\Sigma^0_1(\mathbb{R}^3)$. In view of (2.18), we obtain from Lemma 4.1 that

$$\begin{aligned} \|\exp(-t\Omega S)\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}} &\leq \|\cos(t\Omega i\mathcal{R}_3)\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}} + \|\sin(t\Omega i\mathcal{R}_3)\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}}\|R\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}} \quad (4.3)\\ &\leq C\|\exp(t\Omega\mathcal{R}_3)\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}} + C\|\exp(t\Omega\mathcal{R}_3)\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}}\|R\|_{\mathcal{M}_{p,\mu}\to\mathcal{M}_{p,\mu}}\\ &\leq C(1 + t|\Omega|)^2, \quad (4.4)\end{aligned}$$

which is equivalent to (4.2). □

Lemma 4.4. *Let $n = 3$, $\Omega \in \mathbb{R}$, $1 < q_1 \leq q_2 < \infty, 0 \leq \mu < 3$, $\eta_i = \frac{3-\mu}{q_i}$ $(i = 1, 2)$ and let $\beta \in (\{0\} \cup \mathbb{N})^3$ be a multi-index. There exists $C_1 > 0$ (independent of Ω) such that*

$$\|\partial_x^\beta \mathcal{G}(t)f\|_{q_2,\mu} \leq C_1(1 + t|\Omega|)^2 t^{-\frac{\eta_1-\eta_2}{2}-\frac{|\beta|}{2}}\|f\|_{q_1,\mu}, \tag{4.5}$$

for all $f \in \mathcal{M}^\sigma_{q_1,\mu}(\mathbb{R}^3)$ and $t > 0$.

Proof. In view of (2.17), we can use Lemmas 4.2 and 4.3 to estimate

$$\begin{aligned} \left\|\partial_x^\beta \mathcal{G}(t)f\right\|_{q_2,\mu} &= \left\|\partial_x^\beta e^{t\Delta}\left[e^{-t\Omega S}f\right]\right\|_{q_2,\mu}\\ &\leq Ct^{-\frac{\eta_1-\eta_2}{2}-\frac{|\beta|}{2}}\|e^{-t\Omega S}f\|_{q_1,\mu}\\ &\leq C_1(1 + t|\Omega|)^2 t^{-\frac{\eta_1-\eta_2}{2}-\frac{|\beta|}{2}}\|f\|_{q_1,\mu},\end{aligned}$$

which gives (4.5). □

4.2. Bilinear estimates

From now on, we denote

$$B(u, v)(x, t) := \int_0^t \nabla \cdot \mathcal{G}(t - s)\mathbb{P}(u \otimes v)(s)ds, \tag{4.6}$$

where the above integral should be understood in the sense of Bochner.

Lemma 4.5. *Let $T > 0$, $1 < q' < p < q < \infty, 0 \leq \mu < 3$ and $\mu \geq 3 - p$. There exists a constant $C_2 > 0$ such that*

$$\|B(u, v)\|_{H_{q,T}} \leq C_2\,(1 + T|\Omega|)^2 T^{\frac{1}{2}-\frac{3-\mu}{2p}}\|u\|_{H_{q,T}}\|v\|_{H_{q,T}}, \tag{4.7}$$

for all $u, v \in H_{q,T}$, where C_2 is independent of T and the rotation speed Ω.

Proof. Note that $\alpha = \frac{3-\mu}{p} - \frac{3-\mu}{q} < 1$. From the semigroup estimate (4.5) with $(|\beta|, q_2, q_1) = (1, q, \frac{q}{2})$ and Hölder inequality (2.3), we get

$$\begin{aligned} \|\nabla \cdot \mathcal{G}(t - s)[\mathbb{P}(u \otimes v)(s)]\|_{q,\mu} &\leq C\,(1 + (t - s)|\Omega|)^2(t - s)^{-\frac{1}{2}\left(\frac{3-\mu}{q/2}-\frac{3-\mu}{q}\right)-\frac{1}{2}}\|(u \otimes v)(\cdot, s)\|_{\frac{q}{2},\mu}\\ &\leq C\,(1 + (t - s)|\Omega|)^2(t - s)^{\frac{1}{2}-\frac{3-\mu}{2p}+\frac{\alpha}{2}-1}\|u(\cdot, s)\|_{q,\mu}\,\|v(\cdot, s)\|_{q,\mu}. \quad (4.8)\end{aligned}$$

It follows from (4.8) that

$$\begin{aligned}\|B(u,v)(\cdot,t)\|_{q,\mu} &\le \int_0^t \|\nabla\cdot\mathcal{G}(t-s)\mathbb{P}(u\otimes v)(s)\|_{q,\mu}ds\\ &\le K(t)\sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(\cdot,t)\|_{q,\mu}\sup_{0<t<T} t^{\frac{\alpha}{2}}\|v(\cdot,t)\|_{q,\mu} \qquad (4.9)\\ &\le K(t)\|u\|_{H_{q,T}}\|v\|_{H_{q,T}}, \qquad (4.10)\end{aligned}$$

where

$$\begin{aligned}K(t) &= C\int_0^t (1+(t-s)|\Omega|)^2(t-s)^{\frac{1}{2}-\frac{3-\mu}{2p}+\frac{\alpha}{2}-1}s^{-\alpha}ds\\ &\le C(1+t|\Omega|)^2\int_0^t (t-s)^{\frac{1}{2}-\frac{3-\mu}{2p}+\frac{\alpha}{2}-1}s^{-\alpha}ds\\ &= C\,(1+t|\Omega|)^2 t^{\frac{1}{2}-\frac{3-\mu}{2p}-\frac{\alpha}{2}}\int_0^1 (1-z)^{\frac{1}{2}-\frac{3-\mu}{2p}+\frac{\alpha}{2}-1}z^{-\alpha}dz\\ &\le C(1+T|\Omega|)^2T^{\frac{1}{2}-\frac{3-\mu}{2p}}t^{-\frac{\alpha}{2}}. \qquad (4.11)\end{aligned}$$

The estimates (4.10) and (4.11) give us

$$\sup_{0<t<T} t^{\frac{\alpha}{2}}\|B(u,v)(\cdot,t)\|_{q,\mu} \le C(1+T|\Omega|)^2T^{\frac{1}{2}-\frac{3-\mu}{2p}}\|u\|_{H_{q,T}}\|v\|_{H_{q,T}}. \qquad (4.12)$$

Proceeding similarly to the proof of (4.9), one also obtains

$$\begin{aligned}\sup_{0<t<T}\|B(u,v)(\cdot,t)\|_{p,\mu} &\le \sup_{0<t<T}\left(C(1+t|\Omega|)^2\int_0^t (t-s)^{\frac{1}{2}-\frac{3-\mu}{2p}+\frac{\alpha}{2}-1}s^{-\frac{\alpha}{2}}ds\right)\\ &\quad\cdot\left(\sup_{0<t<T}\|u(\cdot,t)\|_{p,\mu}\right)\left(\sup_{0<t<T} t^{\frac{\alpha}{2}}\|v(\cdot,t)\|_{q,\mu}\right) \qquad (4.13)\\ &\le C\,(1+T|\Omega|)^2T^{\frac{1}{2}-\frac{3-\mu}{2p}}\|u\|_{H_{q,T}}\|v\|_{H_{q,T}}. \qquad (4.14)\end{aligned}$$

The estimate (4.7) follows from (4.12) and (4.14). □

4.3. *Proof of Theorem 3.1*

It follows from (4.5) that (changing C_1 if necessary)

$$\|\mathcal{G}(t)u_0\|_{H_{q,T}} \le C_1(1+T|\Omega|)^2\|u_0\|_{p,\mu} \qquad (4.15)$$

for all $u_0 \in \mathcal{M}^{\sigma}_{p,\mu}$. Furthermore, Lemma 4.5 implies that $B(\cdot,\cdot)$ is bi-continuous in $H_{q,T}$ with

$$\|B\|_{H_{q,T}\times H_{q,T}\to H_{q,T}} \le C_2(1+T|\Omega|)^2T^{\frac{1}{2}-\frac{3-\mu}{2p}}. \qquad (4.16)$$

We claim that there is $T_\Omega > 0$ such that Φ defined by

$$\Phi(u)(t) := \mathcal{G}(t)u_0 + B(u,u)(t) \qquad (4.17)$$

is a contraction in $\mathcal{B}_{2\gamma} = \{u \in H_{q,T_\Omega};\ \|u\|_{H_{q,T_\Omega}} \le 2\gamma_\Omega\}$ endowed with $\mathcal{Z}(u,\tilde{u}) = \|u-\tilde{u}\|_{H_{q,T_\Omega}}$, where

$$\gamma_\Omega = C_1(1+T_\Omega|\Omega|)^2\|u_0\|_{p,\mu}.$$

For that, we choose $T_\Omega > 0$ such that

$$4\gamma_\Omega C_2(1+T_\Omega|\Omega|)^2T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}} = 4C_1C_2(1+T_\Omega|\Omega|)^4T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}}\|u_0\|_{p,\mu} < 1. \qquad (4.18)$$

Using the bilinearity of $B(\cdot,\cdot)$ and (4.16), we estimate

$$\begin{aligned}\|\Phi(u)-\Phi(\tilde{u})\|_{H_{q,T_\Omega}} &\le C_2(1+T_\Omega|\Omega|)^2T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}}(\|u\|_{H_{q,T_\Omega}}+\|\tilde{u}\|_{H_{q,T_\Omega}})\|u-\tilde{u}\|_{H_{q,T_\Omega}} \qquad (4.19)\\ &\le 4\gamma_\Omega C_2(1+T_\Omega|\Omega|)^2T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}}\|u-\tilde{u}\|_{H_{q,T_\Omega}}, \qquad (4.20)\end{aligned}$$

for all $u, \tilde{u} \in \mathcal{B}_{2\gamma_\Omega}$. Since $\Phi(0) = \mathcal{G}(t)u_0$, taking $\tilde{u} = 0$ in (4.19) and using (4.15), we get

$$\|\Phi(u)\|_{H_{q,T_\Omega}} \leq \|\mathcal{G}(t)u_0\|_{H_{q,T_\Omega}} + \|\Phi(u) - \mathcal{G}(t)u_0\|_{H_{q,T_\Omega}} \tag{4.21}$$

$$\leq C_1(1 + T_\Omega|\Omega|)^2\|u_0\|_{p,\mu} + C_2(1 + T_\Omega|\Omega|)^2 T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}} \|u\|^2_{H_{q,T_\Omega}}$$

$$\leq \gamma_\Omega + \left(4C_2(1 + T_\Omega|\Omega|)^2 T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}} \gamma_\Omega\right)\gamma_\Omega < 2\gamma_\Omega, \tag{4.22}$$

provided $u \in \mathcal{B}_{2\gamma_\Omega}$. Now the desired claim follows from (4.18), (4.20) and (4.22), and so Φ has a fixed point $u \in \mathcal{B}_{2\gamma_\Omega}$ which is a mild solution for (1.1)–(1.3). This solution is unique in $\mathcal{B}_{2\gamma_\Omega}$. Notice that the initial data and radius $2\gamma_\Omega$ can be large, when $\mu > 3 - p$.

Let $u, \tilde{u}$ be two mild solutions with same existence time T_Ω and respective data $u_0, \tilde{u}_0$. The Lipschitz continuity of data-solution map follows from (4.18) and estimate

$$\|u - \tilde{u}\|_{H_{q,T_\Omega}} = \|\Phi(u) - \Phi(\tilde{u})\|_{H_{q,T_\Omega}}$$

$$\leq \|\mathcal{G}(t)(u_0 - \tilde{u}_0)\|_{H_{q,T_\Omega}} + \|B(u,u) - B(\tilde{u},\tilde{u})\|_{H_{q,T_\Omega}}$$

$$\leq C_1(1 + T_\Omega|\Omega|)^2\|u_0 - \tilde{u}_0\|_{p,\mu} + 4\gamma C_2(1 + T_\Omega|\Omega|)^2 T_\Omega^{\frac{1}{2}-\frac{3-\mu}{2p}} \|u - \tilde{u}\|_{H_{q,T_\Omega}}.$$

For $\mu = 3 - p$, it is sufficient to observe that (4.18) holds true for

$$\|u_0\|_{p,\mu} < \delta = \frac{1}{4C_1C_2(1 + T|\Omega|)^4} \leq \frac{1}{4C_1C_2} = \delta_0. \tag{4.23}$$

In the case $\Omega = 0$, we have $\delta = \delta_0$ and (4.18) holds true, regardless of T. Thus we can take $T_0 = \infty$. Also, given $\|u_0\|_{p,\mu} < \delta_0$, there is $\eta > 0$ such that $\|u_0\|_{p,\mu} < \delta \leq \delta_0$ provided that $0 \leq T\,|\Omega| \leq \eta$. This concludes the proof. □

4.4. Proof of Theorem 3.4

Part (i) (*Vanishing angular velocity limit*): we have that u_Ω and w satisfy the respective equations

$$u_\Omega(t) = e^{t(\Delta - S\Omega)}u_0 + \int_0^t \nabla\cdot e^{(t-s)(\Delta - \Omega S)}\mathbb{P}(u_\Omega \otimes u_\Omega)(s)ds \tag{4.24}$$

and

$$w(t) = e^{t\Delta}u_0 + \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}(w \otimes w)(s)ds. \tag{4.25}$$

In view of Remark 3.2 and (4.18), we can take $T_\Omega = T_0 < \infty$ when $|\Omega|$ is small enough and $\mu > 3 - p$. So, we set $T = T_\Omega = T_0$ in this case. Also, for $\mu = 3 - p$, given an arbitrary $T > 0$, there is $\eta > 0$ small enough such that $T \leq T_\Omega < T_0 = \infty$, for all $|\Omega| \leq \eta$.

From existence result (see Theorem 3.1), we have that $\|w\|_{H_{q,T}} \leq \|w\|_{H_{q,T_0}} \leq 2\gamma_0$ and $\|u_\Omega\|_{H_{q,T}} \leq \|u_\Omega\|_{H_{q,T_\Omega}} \leq 2\gamma_\Omega$. Given $f \in \mathcal{M}^\sigma_{p,\mu}$, notice that

$$\sup_{0<t<T}\left\|\left(e^{-t\Omega S} - 1\right)f\right\|_{p,\mu} \leq \sup_{0<t<T}\int_0^t \left\|\frac{d}{d\tau}(e^{-\tau\Omega S}f)\right\|_{p,\mu} d\tau$$

$$\leq |\Omega|\,T \sup_{0<\tau<T}\left\|Se^{-\tau\Omega S}f\right\|_{p,\mu} \tag{4.26}$$

$$\leq C\,|\Omega|\,T(1 + T|\Omega|)^2\,\|f\|_{p,\mu} \to 0, \quad \text{as} \quad \Omega \to 0, \tag{4.27}$$

where, from (4.26) to (4.27), we have used Lemma 4.3 and the continuity of $S = \mathbb{PJP}$ on $\mathcal{M}^\sigma_{p,\mu}$. Subtracting the Eqs. (4.24) and (4.25) we get

$$\|u_\Omega(\cdot,t) - w(\cdot,t)\|_{p,\mu} \leq \left\|(e^{-tS\Omega} - 1)e^{t\Delta}u_0\right\|_{p,\mu} + \left\|\int_0^t (e^{-(t-s)S\Omega} - 1)\nabla\cdot e^{(t-s)\Delta}\mathbb{P}(u_\Omega \otimes u_\Omega)(s)ds\right\|_{p,\mu}$$

$$+ \left\|\int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}\left[(u_\Omega - w)\otimes u_\Omega + w\otimes(u_\Omega - w)\right]ds\right\|_{p,\mu}$$

$$:= I_0(t,\Omega) + I_1(t,\Omega) + I_2(t,\Omega). \tag{4.28}$$

The terms $I_j(t,\Omega)$ can be estimated in the following way:

$$\sup_{0<t<T} I_0(t,\Omega) \leq C\,|\Omega|\,T(1+T|\Omega|)^2 \left\|e^{t\Delta}u_0\right\|_{p,\mu} \leq C\,|\Omega|\,T(1+T|\Omega|)^2 \left\|u_0\right\|_{p,\mu}, \tag{4.29}$$

$$\begin{aligned}\sup_{0<t<T} I_1(t,\Omega) &\leq \int_0^t \left\|(e^{-(t-s)S\Omega}-1)\nabla\cdot e^{(t-s)\Delta}\mathbb{P}(u_\Omega\otimes u_\Omega)(s)\right\|_{p,\mu} ds \\ &\leq C\,|\Omega|\,T(1+T|\Omega|)^2 \int_0^t \left\|\nabla\cdot e^{(t-s)\Delta}\mathbb{P}(u_\Omega\otimes u_\Omega)(s)\right\|_{p,\mu} ds \\ &\leq C\,|\Omega|\,T(1+T|\Omega|)^2 C_2(1+T|\Omega|)^2 T^{\frac{1}{2}-\frac{3-\mu}{2p}} \left\|u_\Omega\right\|_{H_{q,T}} \left\|u_\Omega\right\|_{H_{q,T}} \\ &\leq C\left(4\gamma_\Omega^2\right)|\Omega|\,T^{\frac{3}{2}-\frac{3-\mu}{2p}}(1+T|\Omega|)^4,\end{aligned} \tag{4.30}$$

and (proceeding as in (4.13))

$$\begin{aligned}\sup_{0<t<T} I_2(t,\Omega) &\leq C(1+T|\Omega|)^2 \sup_{0<t<T}\int_0^t (t-s)^{\alpha/2-1}\left[\left\|(u_\Omega-w)(\cdot,s)\right\|_{p,\mu}\left(\left\|u_\Omega(\cdot,s)\right\|_{q,\mu}+\left\|w(\cdot,s)\right\|_{q,\mu}\right)\right] ds \\ &\leq 2(\gamma_\Omega+\gamma_0)C(1+T|\Omega|)^2 \sup_{0<t<T}\int_0^t (t-s)^{\alpha/2-1}s^{-\alpha/2}\left\|(u_\Omega-w)(\cdot,s)\right\|_{p,\mu} ds.\end{aligned} \tag{4.31}$$

In view of (4.29)–(4.31), afterwards applying $\sup_{0<t<T}$ and $\limsup_{\Omega\to 0}$ in (4.28), we obtain

$$\begin{aligned}&\limsup_{\Omega\to 0}\left(\sup_{0<t<T}\left\|u_\Omega(\cdot,t)-w(\cdot,t)\right\|_{p,\mu}\right) \\ &\leq \limsup_{\Omega\to 0}\left(\sup_{0<t<T} I_0(t,\Omega)\right)+\limsup_{\Omega\to 0}\left(\sup_{0<t<T} I_1(t,\Omega)\right)+\limsup_{\Omega\to 0}\left(\sup_{0<t<T} I_2(t,\Omega)\right) \\ &= 0+0+4\gamma_0 C\left(\sup_{0<t<T}\int_0^t (t-s)^{\alpha/2-1}s^{-\alpha/2}ds\right)\limsup_{\Omega\to 0}\left(\sup_{0<t<T}\left\|u_\Omega(\cdot,t)-w(\cdot,t)\right\|_{p,\mu}\right) \\ &\leq 4\gamma_0 C_2 T^{\frac{1}{2}-\frac{3-\mu}{2p}}\limsup_{\Omega\to 0}\left(\sup_{0<t<T}\left\|u_\Omega(\cdot,t)-w(\cdot,t)\right\|_{p,\mu}\right),\end{aligned}$$

which implies that

$$\lim_{\Omega\to 0}\left(\sup_{0<t<T}\left\|u_\Omega(\cdot,t)-w(\cdot,t)\right\|_{p,\mu}\right)=0,$$

because $4\gamma_0 C_2 T^{\frac{1}{2}-\frac{3-\mu}{2p}} \leq 4\gamma_0 C_2 T_0^{\frac{1}{2}-\frac{3-\mu}{2p}} < 1$ (see (4.18)), as required. □

Part (ii) (*Approximate local-in-time self-similarity*): if u_0 is homogeneous of degree -1 and $w(x,t)\in H_{q,\infty}$ verifies (4.25), then it is not difficult to check that $w_\lambda(x,t)=\lambda w(\lambda x,\lambda^2 t)$ also verifies (4.25), for each fixed $\lambda>0$. Since $\mu=3-p$, we have that (3.5) holds true, and then

$$\left\|w_\lambda\right\|_{H_{q,\infty}}=\left\|w\right\|_{H_{q,\infty}}\leq 2\gamma_0.$$

From uniqueness assertions of Theorem 3.1(ii), we obtain $w_\lambda=w$ in $H_{q,\infty}$, and then w is self-similar. In view of (3.7), it follows that u_Ω is approximately self-similar in $L^\infty_{\mathrm{loc}}(0,\infty;\mathcal{M}_{p,\mu})$ for small values of $|\Omega|$. □

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