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Symmetry, bifurcation and stacking of the central configurations of the planar 1 + 4 body problem

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Abstract In this work we are interested in the central configurations of the planar 1 + 4 body problem where the satellites have different infinitesimal masses and two of them are diametrically opposite in a circle. We can think of this problem as a stacked central configuration too. We show that the configurations are necessarily symmetric and the other satellites have the same mass. Moreover we prove that the number of central configurations in this case is in general one, two or three and, in the special case where the satellites diametrically opposite have the same mass, we prove that the number of central configurations is one or two and give the exact value of the ratio of the masses that provides this bifurcation.

Keywords Central configurations · Planar coorbital satellites · Bifurcation · Symmetry · Stacked central configurations

1 Introduction

The N-body problem concerns the study of the dynamics of N point masses subject to their mutual Newtonian gravitational interaction. If N > 2, it is impossible to obtain a general solution to this problem because of its non-integrability; hence, particular solutions are very important in this case. Central configurations are the configurations for which the total Newtonian acceleration of every body is equal to a constant multiplied by the position vector of this body with respect to the center of mass of the configuration. One of the reasons why central configurations are interesting is that they allow us to obtain explicit homographic solutions of the N-body problem i. e. motion where the configuration of the system changes its size but keeps its shape. Secondly, they arise as the limiting configuration of a total collapse. The papers dealing with central configurations, giving the number of central configurations and studying their symmetry, stability, stacking properties, etc. A stacked central configuration is

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a central configuration that contains others central configurations by dropping some bodies. Some references on this theme are Hagihara (1970), Saari (1980, 2005) and Wintner (1941).

This work deals with central configurations of the planar 5-body problem, in which there is one dominant mass and four infinitesimal masses, called satellites, on a plane. The planar 1 + n body problem was treated by Maxwell (1859) who was trying to construct a model for Saturn's rings. There are many other contributions in the literature for this problem. Considering satellites with equal masses, Casasayas et al. (1994) improved a previous result of Glen Hall in an unpublished work proving that the regular polygon is the only central configuration if $n \ge e^{73}$. Cors et al. (2004) proved that there are only three symmetric central configurations of the 1 + 4 body problem and obtained numerically evidences that there is only one central configuration if $n \ge 9$ and every central configuration is symmetric with respect to a straight line. This last statement was proven by Albouy and Fu (2009) in the case 1 + 4 showing that all central configurations of four identical satellites are symmetric.

In a recent paper, Oliveira and Cabral (2012) worked with stacked planar central configuration of the 1 + n body problem in two cases: First, adding one satellite to a central configuration with different satellites and the second, adding two satellites considering equal all infinitesimal masses. Renner and Sicardy (2004) obtained results about finding the infinitesimal masses that make up a central configuration, if a configuration of the co-orbital satellites is fixed. They also studied the linear stability of this configuration. Corbera et al. (2011) considering the 1 + 3 body problem, found two different classes exhibiting symmetric and non-symmetric configurations. When two infinitesimal masses are equal, they provide evidence that the number of central configurations varies from five to seven.

In this work we are interested in the central configurations of the planar 1 + 4 body problem where the satellites have different masses. The satellites lie on a circle centered at the big mass and we treat the case where two of them are diametrically opposite. Observe that this problem is a stacked central configuration too since the collinear 1 + 2 configuration is a central one.

2 Preliminaries

We develop in this section the formulation of this well known problem. Basically the reference is our previous work (Oliveira and Cabral 2012). More details can be found in (Casasayas et al. 1994). Consider N point masses, m_1, \ldots, m_N , in the plane subject to their mutual Newtonian gravitational interaction. Let $q_i \in \mathbb{R}^2$ be the position of the mass m_i . The equations of motion in an inertial reference frame with origin at the center of mass are given by

$$m_i \ddot{q_i} = \sum_{\substack{j=1 \ j \neq i}}^N m_i m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad i = 1, \dots, N.$$

Let $M = diag\{m_1, m_1, \dots, m_N, m_N\}$ be the matrix of masses and let $q = (q_1, \dots, q_N)$, $q_i \in \mathbb{R}^2$ be the position vector. The equations above become

$$M\ddot{q} = \frac{\partial V}{\partial q},$$

where $V(q_1, ..., q_N) = \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|}$ is the Newtonian potential.

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A non-collision configuration $q = (q_1, ..., q_N)$ with $\sum_{i=1}^N m_i q_i = 0$ is a *central configuration* if there exists a positive constant λ such that

$$M^{-1}\frac{\partial V}{\partial q} = \lambda q.$$

Let $q(\epsilon) = (q_0(\epsilon), q_1(\epsilon), \dots, q_n(\epsilon))$ be a central configuration of the planar N body problem with masses $m_0 = 1$ and $m_i = \mu_i \epsilon$, $i = 1, \dots, n$, where n = N - 1. We say that $q = (q_0, q_1, q_2, \dots, q_n)$ is a *central configuration of the planar* 1 + n body problem if there exists $\lim_{\epsilon \to 0} q(\epsilon)$ and this limit is equal to q.

In this work we deal only with non-coalescent central configuration of the planar 1 + n body problem, i.e. we exclude the case where two small bodies coincide at the limit [see Moeckel (1997)].

In all central configurations of the planar 1 + n body problem the *n* small bodies, called satellites, lie on a circle centered at the big mass (Casasayas et al. 1994), i.e. they are coorbital. Since we are interested in central configurations modulo rotations and homothetic transformations, we can assume that the circle has radius 1 and $q_1 = (1, 0)$.

We take as coordinates the angles θ_i between two consecutive satellites. See Casasayas et al. (1994) for details. In these coordinates the configuration space is the simplex

$$\Delta = \{ \theta = (\theta_1, \dots, \theta_n); \quad \sum_{i=1}^n \theta_i = 2\pi, \theta_i > 0, \quad i = 1, \dots, n \}$$

and the equations characterizing the central configurations of the planar 1+n body problem are

$$\mu_{2} f(\theta_{1}) + \mu_{3} f(\theta_{1} + \theta_{2}) + \dots + \mu_{n} f(\theta_{1} + \theta_{2} + \dots + \theta_{n-1}) = 0,$$

$$\mu_{3} f(\theta_{2}) + \mu_{4} f(\theta_{2} + \theta_{3}) + \dots + \mu_{1} f(\theta_{2} + \theta_{3} + \dots + \theta_{n}) = 0,$$

$$\mu_{4} f(\theta_{3}) + \mu_{5} f(\theta_{3} + \theta_{4}) + \dots + \mu_{2} f(\theta_{3} + \theta_{4} + \dots + \theta_{n} + \theta_{1}) = 0,$$

$$\dots$$

$$\mu_{n} f(\theta_{n-1}) + \dots + \mu_{n-2} f(\theta_{n-1} + \theta_{n} + \theta_{1} + \dots + \theta_{n-3}) = 0,$$

$$\mu_{1} f(\theta_{n}) + \mu_{2} f(\theta_{n} + \theta_{1}) + \dots + \mu_{n-1} f(\theta_{n} + \theta_{1} + \theta_{2} + \dots + \theta_{n-2}) = 0,$$

$$\theta_{1} + \dots + \theta_{n} = 2\pi,$$

$$\theta_{i} > 0,$$
where
$$f(x) = \sin(x) \left(1 - \frac{1}{8|\sin^{3}(x/2)|} \right).$$

$$(1)$$

Proposition 1 Every solution $(\theta_1, \ldots, \theta_n)$ of the system (1) is a non-coalescent central configuration of the planar 1 + n body problem associated to the mass parameters μ_1, \ldots, μ_n .

Proof See Casasayas et al. (1994).

The following results exhibit the main properties of the function f. Their proof can be found in Albouy and Fu (2009).

Lemma 1 The function

$$f(x) = \sin(x) \left(1 - \frac{1}{8|\sin^3(x/2)|} \right), \quad x \in (0, 2\pi)$$

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satisfies:

- (i) $f(\pi/3) = f(\pi) = f(5\pi/3) = 0;$ (ii) $f(\pi - x) = -f(\pi + x), \forall x \in (0, \pi);$ (iii) $f'(x) = \cos(x) + \frac{3 + \cos(x)}{16|\sin^3(x/2)|} \ge f'(\pi) = -7/8, \text{ for all } x \in (0, 2\pi);$ (iv) $f'''(x) > 0, \text{ for all } x \in (0, 2\pi);$
- (v) In $(0, \pi)$ there is a unique critical point θ_c of f such that $\theta_c > 3\pi/5$, $f'(\theta) > 0$ in $(0, \theta_c)$ and $f'(\theta) < 0$ in (θ_c, π) .

Lemma 2 Consider four points t_1^L , t_1^R , t_2^L , t_2^R such that $0 < t_1^L < t_2^L < \theta_c < t_2^R < t_1^R < 2\pi$, $f(t_1^L) = f(t_1^R) = f_1$ and $f(t_2^L) = f(t_2^R) = f_2$. Then $t_2^L + t_2^R < t_1^L + t_1^R$.

Corollary 1 Consider $0 < t_1 < \theta_c < t_2 < 2\pi$. If $f(t_1) \ge f(t_2)$ then $t_1 + t_2 > 2\theta_c > 6\pi/5$.

3 Main results

We now consider the planar problem of 1 + 4 bodies, where the four satellites do not necessarily have the same masses. The goal is to find all central configurations with two diametrically opposite satellites. We call them the collinear satellites. See Fig. 2. In this way we have a central configuration of the planar 1+2 body problem in which the satellites and the massive body are collinear. Hence we also get stacked central configurations as introduced by Hampton (2005). Since $f(x) = -f(2\pi - x)$, in the case n = 4, the system (1) becomes

$$\mu_{2}f_{1} + \mu_{3}f_{12} = \mu_{4}f_{4},$$

$$\mu_{3}f_{2} + \mu_{4}f_{23} = \mu_{1}f_{1},$$

$$\mu_{4}f_{3} + \mu_{1}f_{34} = \mu_{2}f_{2},$$

$$\mu_{1}f_{4} + \mu_{2}f_{14} = \mu_{3}f_{3},$$

$$\theta_{1} + \theta_{2} + \theta_{3} + \theta_{4} = 2\pi,$$
(2)

where

$$f_i = f(\theta_i)$$
 and $f_{ij} = f(\theta_i + \theta_j)$.

First we consider the case where the two collinear satellites are arranged consecutively in the circle, like the case (ii) in Fig. 2. The next result shows that it is impossible to have a central configuration like that.

Theorem 1 Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be a non-coalescent central configuration of the planar 1 + 4 body problem associated to mass parameters $\mu_1, \mu_2, \mu_3, \mu_4$. Then the massive body and any two consecutive satellites cannot be collinear, *i.* e. $\theta_i \neq \pi$ for all i = 1, 2, 3, 4.

Proof Suppose without loss of generality that $\theta_4 = \theta_1 + \theta_2 + \theta_3 = \pi$. The system (2) becomes

$$\mu_2 f_1 = \mu_3 f(\pi + \theta_3), \tag{3}$$

$$\mu_3 f_2 = \mu_4 f(\pi + \theta_1) + \mu_1 f_1, \tag{4}$$

$$\mu_4 f_3 + \mu_1 f_{34} = \mu_2 f_2, \tag{5}$$

$$\mu_3 f_3 = \mu_2 f(\pi + \theta_1), \tag{6}$$

Suppose that $f(\pi + \theta_1) \ge 0$. Then $\pi + \theta_1 \ge 5\pi/3$ and we get $\theta_1 \ge 2\pi/3$ and $\theta_2 + \theta_3 \le \pi/3$. So $\theta_3 < \pi/3$ and $f_3 < 0$ hence Eq. (6) is impossible. Therefore $f(\pi + \theta_1) < 0$. Analogously $f(\pi + \theta_3) < 0$.

From (3) and (6), we get f_1 , $f_3 < 0$. So $\theta_1 < \pi/3$, $\theta_3 < \pi/3$ and consequently $\theta_2 > \pi/3$ and $f_2 > 0$. Hence the right side of (4) is negative and its left side is positive. This concludes the proof.

The remaining results concern the case (i) in Fig. 2 namely the collinear satellites are not consecutive in the circle. First we prove that the configuration is symmetric and the other satellites are identical.

Theorem 2 Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be a non-coalescent central configuration of the planar 1 + 4body problem associated to mass parameters $\mu_1, \mu_2, \mu_3, \mu_4$. Suppose that the massive body and the satellites with masses μ_1 and μ_3 are collinear, i.e. $\theta_1 + \theta_2 = \pi = \theta_3 + \theta_4$. Then $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$, i.e. the configuration is symmetric. Furthermore the other satellites have the same mass $\mu_2 = \mu_4$.

Proof Since $f(\pi) = 0$, we get

 $\mu_2 f_1 = \mu_4 f_4, \tag{7}$

$$\mu_3 f_2 + \mu_4 f_{23} = \mu_1 f_1, \tag{8}$$

$$\mu_4 f_3 = \mu_2 f_2, \tag{9}$$

$$\mu_1 f_4 + \mu_2 f_{14} = \mu_3 f_3. \tag{10}$$

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Hence, by (7) and (9)

$$f_1 f_3 = f_2 f_4. \tag{11}$$

From (7), $f_1 = 0$ if and only if $f_4 = 0$. As the only root in $(0, \pi)$ of f is $\pi/3$, then if $f_1 = 0$ or $f_4 = 0$ we get $\theta_1 = \theta_4 = \pi/3$. Analogously if $f_2 = 0$ or $f_3 = 0$, then $\theta_2 = \theta_3 = \pi/3$. Therefore we will suppose that $f_i \neq 0$, i = 1, 2, 3, 4 or equivalently $\theta_i \neq \pi/3, 2\pi/3$.

From (11) and by hypothesis $\theta_2 = \pi - \theta_1$, $\theta_3 = \pi - \theta_4$ we have

$$l(\theta_1) = l(\theta_4),\tag{12}$$

where $l(x) = \frac{f(x)}{f(\pi - x)}, x \neq 2\pi/3.$

Note that by (9) f_2 , f_3 have the same sign. Since f is negative in $(0, \pi/3)$ and positive in $(\pi/3, \pi)$ then $0 < \theta_2, \theta_3 < \pi/3$ or $\pi/3 < \theta_2, \theta_3 < \pi$. But $\theta_1 + \theta_2 = \theta_3 + \theta_4 = \pi$, hence $2\pi/3 < \theta_1, \theta_4 < \pi$ or $0 < \theta_1, \theta_4 < 2\pi/3$. So by (12) it is sufficient to show that $l|_{(0,2\pi/3)}$ and $l|_{(2\pi/3,\pi)}$ are injective to prove $\theta_1 = \theta_4$.

We have

$$l'(x) = \frac{f'(x)f(\pi - x) + f(x)f'(\pi - x)}{(f(\pi - x))^2}$$

If $x \in (2\pi/3, \pi)$ then f'(x) < 0, $f(\pi - x) < 0$, f(x) > 0 and $f'(\pi - x) > 0$. So l'(x) > 0 and hence $l|_{(2\pi/3,\pi)}$ is injective.

Let $p(x) = f'(x)f(\pi - x) + f(x)f'(\pi - x)$ be the numerator of l'(x).

$$p'(x) = f''(x)f(\pi - x) - f(x)f''(\pi - x).$$

Suppose that $0 < x < \pi/2$. We get $x < \pi - x$ and consequently $f''(x) < f''(\pi - x)$. If $f(x) \ge f(\pi - x)$ Corollary 1 gives $\pi = x + (\pi - x) > 2\theta_c > 6\pi/5$. So

$$f(x) < f(\pi - x).$$

As $f''(\pi - x) < 0$ and $f(\pi - x) > 0$ we have

$$f(x)f''(\pi - x) > f(\pi - x)f''(\pi - x) > f(\pi - x)f''(x).$$

So p'(x) < 0 in $(0, \pi/2)$. Since $p(x) = p(\pi - x)$, if $x \in (\pi/2, \pi)$ then p'(x) > 0. Therefore $x = \pi/2$ is the point of minimum of p and consequently

$$p(x) \ge p(\pi/2) = 2f'(\pi/2)f(\pi/2) > 0.$$

Thus l'(x) > 0 and hence $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$.

Now if $\mu_2 \neq \mu_4$, by (7), (8), (9) and (10) we get $f(\theta_1) = f(\theta_2) = f(2\theta_1) = f(2\theta_2) = 0$. But this is impossible because the roots of f in $(0, 2\pi)$ are $\pi/3$, π and $5\pi/3$.

In the next result we count the central configurations of this problem in the general case. By the last theorem the configurations are symmetric and two satellites have the same infinitesimal masses. The next theorem also shows that an additional equality of some infinitesimal masses are equivalent to the existence of special configurations as a square and a kite.

Theorem 3 Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be a non-coalescent central configuration of the planar 1 + 4 body problem associated to mass parameters $\mu_1, \mu_2, \mu_3, \mu_4$. Suppose that $\theta_1 + \theta_2 = \theta_3 + \theta_4 = \pi$. Then for all values of the mass parameters the number of classes of central configuration is one, two or three. Moreover the square $(\pi/2, \pi/2, \pi/2, \pi/2)$ is a central configuration if and only if $\mu_3 = \mu_1$ and the kite $(2\pi/3, \pi/3, \pi/3, 2\pi/3)$ is a central configuration if and only if $\mu_1 = \mu_2$

Proof By Theorem 2 we know that $\mu_2 = \mu_4$, $\theta_1 = \theta_4$, $\theta_2 = \theta_3 = \pi - \theta_1$. So the Eqs. (7) and (9) are redundant and (8) and (10) are equivalent to each other and they become

$$f(\theta_1) + \frac{\mu_2}{\mu_1} f(2\theta_1) + \frac{\mu_3}{\mu_1} f(\pi + \theta_1) = 0.$$
(13)

We must see how many roots in $(0, \pi)$ has the function

$$g(x) = f(x) + \frac{\mu_2}{\mu_1} f(2x) + \frac{\mu_3}{\mu_1} f(\pi + x).$$

It is easy to see that $g(x) \to -\infty$ as $x \to 0^+$ and $g(x) \to +\infty$ as $x \to \pi^-$. So for any $\mu_1, \mu_2, \mu_3 > 0$ there is at least one solution of g(x) = 0 in $(0, \pi)$. Moreover, since $g'''(x) = f'''(x) + 8\frac{\mu_2}{\mu_1}f'''(2x) + \frac{\mu_3}{\mu_1}f'''(\pi + x) > 0$, there are at most 3 roots of g in $(0, \pi)$.

Observe that, by Lemma 1,

$$g(\pi/2) = f(\pi/2) + \frac{\mu_2}{\mu_1} f(\pi) + \frac{\mu_3}{\mu_1} f(\pi + \pi/2)$$

= $f(\pi/2) + \frac{\mu_3}{\mu_1} f(\pi + \pi/2)$
= $f(\pi/2) - \frac{\mu_3}{\mu_1} f(\pi - \pi/2)$
= $f(\pi/2) \left(1 - \frac{\mu_3}{\mu_1}\right).$

Hence $g(\pi/2) = 0$ if and only if $\mu_3 = \mu_1$. Likewise

$$g(2\pi/3) = f(2\pi/3) + \frac{\mu_2}{\mu_1} f(4\pi/3) + \frac{\mu_3}{\mu_1} f(5\pi/3)$$

= $f(2\pi/3) - \frac{\mu_2}{\mu_1} f(2\pi/3)$
= $f(2\pi/3) \left(1 - \frac{\mu_2}{\mu_1}\right).$

So $g(2\pi/3) = 0$ if and only if $\mu_2 = \mu_1$.

Remark 1 Theorem 3 shows that if $\mu_1 = \mu_2 = \mu_4$ then the kite $(2\pi/3, \pi/3, \pi/3, 2\pi/3)$ is a central configuration for all μ_3 and if $\mu_1 = \mu_2 = a$ and $\mu_2 = \mu_4 = b$, then for all a, b the square $(\pi/2, \pi/2, \pi/2, \pi/2)$ is a central configuration.

The special case where the collinear satellites have the same mass is completely treated in the next theorem. We have now two parameters of masses, $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$. The ratio μ_2/μ_1 provides a parameter for bifurcation in the number of central configuration of this problem. We give its exact value in the theorem.

Theorem 4 Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be a non-coalescent central configuration of the planar 1 + 4body problem associated to mass parameters $\mu_1, \mu_2, \mu_3, \mu_4$. Assume that $\theta_1 = \theta_4, \theta_2 = \theta_3 = \pi - \theta_1$. Suppose that $\mu_1 = \mu_3$. If $\mu_2/\mu_1 \le \frac{3\sqrt{2}}{7}$ there is a unique central configuration: the square $\theta_i = \pi/2$, i = 1, 2, 3, 4. If $\mu_2/\mu_1 > \frac{3\sqrt{2}}{7}$ there are two central configurations: the square and the kite $(\theta_1, \pi - \theta_1, \pi - \theta_1, \theta_1)$ where $\theta_1 \in (\pi/6, \pi/2)$. In fact the function mapping $\mu_2/\mu_1 \in (\frac{3\sqrt{2}}{7}, +\infty)$ to $\theta_1 \in (\pi/6, \pi/2)$ in the kite configuration is a bijective function.

Proof Let $a = \mu_2/\mu_1$. The central configurations are determined by the equation

$$f(\theta_1) + af(2\theta_1) + f(\pi + \theta_1) = 0, \tag{14}$$

with $\theta_1 \in (0, \pi)$. Again, consider the function $g : (0, \pi) \to \mathbb{R}$ given by

$$g(x) = f(x) + af(2x) + f(\pi + x).$$

Since g'''(x) > 0, then $g''(x) = f''(x) + 4af''(2x) + f''(\pi + x)$ is increasing. Moreover $g''(\pi/2) = 0$ because $f''(\pi - x) = -f''(\pi + x)$, so $x = \pi/2$ is the minimal point of g'(x) in $(0, \pi)$.

We obtain

$$g'(\pi/2) = f'(\pi/2) + 2af'(\pi) + f'(3\pi/2)$$
(15)

$$= 2f'(\pi/2) + 2af'(\pi)$$
(16)

$$= 2\left(\frac{3\sqrt{2}}{8} - \frac{7a}{8}\right) \tag{17}$$

$$=\frac{1}{4}(3\sqrt{2}-7a)$$
(18)

So if $a \leq \frac{3\sqrt{2}}{7}$ then $g'(x) \geq g'(\pi/2) \geq 0$ and consequently g has only one root in $(0, \pi)$. As $x = \pi/2$ is always a root of g, then we have only the square $(\pi/2, \pi/2, \pi/2, \pi/2)$.

If $a > \frac{3\sqrt{2}}{7}$ then $g'(\pi/2) < 0$, so g has three roots in $(0, \pi)$, namely $\theta_1^*, \pi/2$ and θ_1^{**} . Since $g(\pi - x) = -g(x)$ then $\theta_1^{**} = \pi - \theta_1^*$ and these roots correspond to the same configuration, the kite $(\theta_1, \pi - \theta_1, \pi - \theta_1, \theta_1)$.

Now we will look for the kite central configuration $(\theta_1, \pi - \theta_1, \pi - \theta_1, \theta_1)$. θ_1 agrees with (14). By the symmetry we can consider $\theta_1 \in (0, \pi/2)$. If $\theta_1 \neq \pi/6$ then (14) is equivalent to

$$a = \frac{-f(\pi + \theta_1) - f(\theta_1)}{f(2\theta_1)}.$$
(19)

If $\theta_1 < \pi/6$ the right-hand side of (19) is negative but the other side is positive. Also observe that $\theta_1 = \pi/6$ does not agree with (14). So we have no solution in $(0, \pi/6]$. Furthermore the right-hand side of (19) satisfies

$$\lim_{\theta_1 \to \pi/6^+} \frac{-f(\pi + \theta_1) - f(\theta_1)}{f(2\theta_1)} = +\infty$$

and

$$\lim_{\theta_1 \to \pi/2^-} \frac{-f(\pi + \theta_1) - f(\theta_1)}{f(2\theta_1)} = \lim_{\theta_1 \to \pi/2^-} \frac{-f'(\pi + \theta_1) - f'(\theta_1)}{2f'(2\theta_1)}$$
$$= \frac{-f'(\pi + \pi/2) - f'(\pi/2)}{2f'(\pi)} = \frac{3\sqrt{2}}{7}$$

Consider then the surjective function $h: (\pi/6, \pi/2) \to (\frac{3\sqrt{2}}{7}, +\infty)$ given by

$$h(x) = \frac{-f(\pi + x) - f(x)}{f(2x)}$$

We claim that h is one-to-one too. In fact it is a decreasing function. To see that we differentiate h and obtain

$$(f(2x))^2 h'(x) = 2f'(2x)(f(x) + f(\pi + x)) - f(2x)(f'(x) + f'(\pi + x)).$$

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The derivative of the right-hand side of the above equation is given by

$$4f''(2x)(f(x) + f(\pi + x)) - f(2x)(f''(x) + f''(\pi + x)) = 4f''(2x)(f(x) - f(\pi - x)) - f(2x)(f''(x) - f''(\pi - x)).$$

The equality follows from Lemma 1. We claim that the above expression is positive. In fact, $x < \pi/2$, hence $\pi - x > x$. It follows that $f''(\pi - x) > f''(x)$ because f'''(x) > 0. By Corollary 1 we have $f(\pi - x) > f(x)$. Moreover f''(2x) < 0 and f(2x) > 0 as $x \in (\pi/6, \pi/2)$. So the statement is true. This shows that the expression for $(f(2x))^2 h'(x)$ takes its maximum value in limit case $x \to \pi/2$. But

$$\lim_{x \to \pi/2} (f(2x))^2 h'(x) = \lim_{x \to \pi/2} (2f'(2x)(f(x) + f(\pi + x))) - f(2x)(f'(x) + f'(\pi + x))) = 0.$$

Therefore $(f(2x))^2 h'(x) < 0$ if $x \in (\pi/6, \pi/2)$ and thus *h* is decreasing in the same interval. The theorem follows.

Remark 2 In Theorems 1, 2 and 4 if we suppose that all satellites have the same mass we obtain Proposition 11 and 12 in Cors et al. (2004).

4 Conclusions

We studied the relative equilibria of the planar 1 + 4 body problem in the case where two satellites are diametrically opposite in the circle centered in the massive body. So these satellites and the big mass are collinear. We show that all central configurations are symmetric kites and a square and the other two satellites have the same mass. Moreover we prove that there are one, two or three such configurations and only in the case where the collinear satellites have different masses it is possible to have three central configurations. If the collinear satellites have equal masses μ_1 and the other satellites have masses μ_2 we gave all relative equilibria and we calculated the value of the ratio μ_2/μ_1 which provides the bifurcation from one to two central configurations.

The collinear configuration with two satellites diametrically opposite is a central configuration of the planar 1 + 2 body problem. So our approach is a study of stacked central configurations too. In this way our results show that adding two new satellites to a collinear 1 + 2 configuration we get a new central configuration if and only if the two new satellites have the same masses, they are put symmetrically and the smaller angle between them and the line of the collinear satellites varies from $\pi/6$ to $\pi/2$.

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