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Results in Mathematics

Solvability of a Class of First Order Differential Operators on the Torus

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Abstract. This paper deals with Gevrey global solvability on the N-dimensional torus ($\mathbb{T}^N \simeq \mathbb{R}^N/2\pi\mathbb{Z}^N$) to a class of nonlinear first order partial differential equations in the form $Lu - au - b\overline{u} = f$, where a, b, and f are Gevrey functions on $\mathbb{T}^N$ and L is a complex vector field defined on $\mathbb{T}^N$. Diophantine properties of the coefficients of L appear in a natural way in our results. Also, we present results in C^∞ context.

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1. Introduction

For $n \geq 1$, let $\mathbb{T}^{n+1} \simeq \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$ be the $(n+1)$-dimensional torus, where the coordinates are denoted by $(x,t) \in \mathbb{T}^n \times \mathbb{T}^1$, with $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$.

Let $s \geq 1$ be a real number. Recall that a complex-valued function f is an s-Gevrey function on $\mathbb{T}^{n+1}$, if f is C^∞ and there exist positive constants C and R such that, for all $\alpha \in \mathbb{Z}_+^{n+1}$ and all $(x,t) \in \mathbb{T}^{n+1}$, one has

$$|\partial^\alpha f(x,t)| \leq CR^{|\alpha|}\alpha!^s.$$

In this paper we will make use of the well-known characterizations of Gevrey functions via Fourier series. A complex-valued function $f(x,t)$ is an s-Gevrey function on $\mathbb{T}^{n+1}$ if f is C^∞ and there exist positive constants C and ϵ such that

$$|\hat{f}(J,k)| \leq Ce^{-\epsilon(\|J\|+|k|)^{1/s}},\ \forall\ (J,k) \in \mathbb{Z}^n \times \mathbb{Z},$$

where $\hat{f}(J,k)$ denotes the (J,k)-coefficient of the Fourier series of $f(x,t)$. Also, $f(x,t)$ is an s-Gevrey function on $\mathbb{T}^{n+1}$ if f is C^∞ and there exist positive constants C, h and ϵ such that

$$|\partial_t^m \hat{f}(J,t)| \le Ch^m m!^s e^{-\epsilon\|J\|^{1/s}}, \quad \forall\, m \in \mathbb{Z}_+,\ \forall\, J \in \mathbb{Z}^n,$$

where $\hat{f}(J,t)$ denotes the J-th coefficient of the partial Fourier series of $f(x,t)$ in the x-variable.

Denote $G^s(\mathbb{T}^{n+1})$ the space of s-Gevrey functions on $\mathbb{T}^{n+1}$. Note that $G^1(\mathbb{T}^{n+1})$ is the space of real-analytic functions on $\mathbb{T}^{n+1}$. For more about Gevrey functions see [20].

For fixed $s \ge 1$, we are interested in the existence of solutions in $G^s(\mathbb{T}^{n+1})$ to a class of first-order partial differential equations given by $Pu = f$, where $f \in G^s(\mathbb{T}^{n+1})$ and $P : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$ has the form

$$Pu = \frac{\partial u}{\partial t} + \sum_{j=1}^{n} C_j \frac{\partial u}{\partial x_j} + Au + B\bar{u}, \tag{1}$$

with $A, B, C_j \in G^s(\mathbb{T}^{n+1})$.

Motivated by [4], we say that P is $s-$*solvable* on $\mathbb{T}^{n+1}$ if for every f in a subspace of $G^s(\mathbb{T}^{n+1})$ of finite codimension there exists $u \in G^s(\mathbb{T}^{n+1})$ such that $Pu = f$ in $\mathbb{T}^{n+1}$. Also, we say that P is $s-$*globally hypoelliptic* if $u \in \mathscr{D}'(\mathbb{T}^{n+1})$ and $Pu \in G^s(\mathbb{T}^{n+1})$ imply that $u \in G^s(\mathbb{T}^{n+1})$.

This paper is a follow-up to the paper [4], where the C^∞ solvability was studied in the two dimensional torus $\mathbb{T}^1 \times \mathbb{T}^1$.

In the case where P is linear, that is, in the case where $B = 0$, the s-solvability problem on $\mathbb{T}^{n+1}$ is treated in [8]. On the other hand, in the case where $B \neq 0$, the operator P is not anymore $\mathbb{C}$-linear; the C^∞ solvability on $\mathbb{T}^2$ was treated in [4]. For related papers see [2,3,6,7,9,11,12,14,16,18].

Our results are linked to Diophantine properties of the coefficients of P.

This work is organized as follows. In Sect. 2, we present a complete characterization of the s-solvability and s-hypoellipticity in the case where P has constant coefficients. In Sect. 3, we deal with the s-solvability for the class of operators with coefficients depending on t given by

$$Pu = \frac{\partial u}{\partial t} - \sum_{j=1}^{n} (p_j(t) + i\lambda_j q(t)) \frac{\partial u}{\partial x_j} - (r(t) + i\delta q(t))u - \alpha q(t)\bar{u},$$

where $p_j, q, r \in G^s(\mathbb{T}^1; \mathbb{R})$, $q \not\equiv 0$, $\delta \in \mathbb{R}$, $\alpha \in \mathbb{C}\setminus\{0\}$, and $\lambda_j \in \mathbb{R}$, $j = 1, \cdots, n$.

Also, we present results in C^∞ context.

2. Operators with Constant Coefficients

In this section we will consider operators P given in the form (1) in the case where P has constant coefficients. More precisely, let $s \ge 1$ and let

$P : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$ be given by

$$Pu = \frac{\partial u}{\partial t} + \sum_{j=1}^{n} C_j \frac{\partial u}{\partial x_j} - Au - B\bar{u}, \tag{2}$$

where $A, B, C_j \in \mathbb{C}$, $j = 1, \cdots, n$. We denote $C = (C_1, \cdots, C_n) \in \mathbb{C}^n$.

For $f \in G^s(\mathbb{T}^{n+1})$ we are interested in finding $u \in G^s(\mathbb{T}^{n+1})$ solution to $Pu = f$ in $\mathbb{T}^{n+1}$. By using Fourier series we can write

$$u(x,t) = \sum_{(J,k)\in\mathbb{Z}^{n+1}} u_{J,k} e^{(J\cdot x+kt)i} \quad \text{and} \quad f(x,t) = \sum_{(J,k)\in\mathbb{Z}^{n+1}} f_{J,k} e^{(J\cdot x+kt)i}.$$

The equation $Pu = f$ leads us to the system

$$\begin{cases} [i(k + C\cdot J) - A]u_{J,k} - B\overline{u_{-J,-k}} & = f_{J,k} \\ \\ -\overline{B}u_{J,k} + [i(k + \overline{C}\cdot J) - \overline{A}]\overline{u_{-J,-k}} & = \overline{f_{-J,-k}} \end{cases} \tag{3}$$

and, consequently,

$$\Delta_{J,k} u_{J,k} = [i(k + \overline{C}\cdot J) - \overline{A}] f_{J,k} + B\overline{f_{-J,-k}}\,, \tag{4}$$

where

$$\begin{aligned} \Delta_{J,k} &= [i(k + C\cdot J) - A][i(k + \overline{C}\cdot J) - \overline{A}] - B\overline{B} \\ &= -|k + C\cdot J|^2 + |A|^2 - |B|^2 - 2i\mathsf{Re}(A(k + \overline{C}\cdot J)). \end{aligned} \tag{5}$$

Hence, in order to find a solution $u \in G^s(\mathbb{T}^{n+1})$ to the equation $Pu = f$ in $\mathbb{T}^{n+1}$ we have to find a sequence $(u_{J,k})$ satisfying (4) and, moreover, such that the series $u(x,t) = \sum_{(J,k)\in\mathbb{Z}^{n+1}} u_{J,k} e^{(J\cdot x+kt)i}$ converges in the G^s topology in $\mathbb{T}^{n+1}$.

Theorem 1. *Let P be given by (2). Then, P is s-solvable on $\mathbb{T}^{n+1}$ if and only if for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that*

$$|\Delta_{J,k}| \geq C_\epsilon e^{-\epsilon(\|J\|+|k|)^{1/s}}, \quad \text{for all } (J,k) \in \mathbb{Z}^n \times \mathbb{Z} \text{ with } \|J\| + |k| \geq C_\epsilon, \tag{6}$$

where $\Delta_{J,k}$ is given by (5).

Proof. First, assume that (6) holds. Hence,

$$\Omega = \{(J,k) \in \mathbb{Z}^n \times \mathbb{Z} : \;\; \Delta_{J,k} = 0\},$$

is a finite set.

Define $\mathscr{F} = \{f \in G^s(\mathbb{T}^{n+1}) : f_{J,k} = 0 \text{ for } (J,k) \in \Omega\}$. Then, $\mathscr{F}$ is a finite codimension subspace of $G^s(\mathbb{T}^{n+1})$. Let $f \in \mathscr{F}$ and let $C_\epsilon > 0$ and $\epsilon > 0$ be such that

$$|\hat{f}(J,k)| \leq C_\epsilon e^{-\epsilon(\|J\|+|k|)^{1/s}}, \; \forall\; (J,k) \in \mathbb{Z}^n \times \mathbb{Z}.$$

By using (6) for $\epsilon/2$ we obtain that the sequence $(u_{J,k})$ given by (4) satisfies

$$\begin{aligned}|u_{J,k}| &\leq C_\epsilon C_{\epsilon/2}^{-1}(|i(k+\overline{C}\cdot J)-\overline{A}|+|B|)e^{-\frac{\epsilon}{2}(\|J\|+|k|)^{1/s}} \\ &\leq \tilde{C}e^{-\frac{\epsilon}{4}(\|J\|+|k|)^{1/s}},\end{aligned}$$

for some $\tilde{C}>0$ and $\|J\|+|k|$ large enough. Hence, P is s-solvable on $\mathbb{T}^{n+1}$ in this case.

Conversely, assume that (6) fails. Then, we can find $\epsilon_0>0$, $C_{\epsilon_0}>0$, and a sequence $(J_\ell,k_\ell)\in\mathbb{Z}^n\times\mathbb{Z}$, satisfying

$$|\Delta_{J_\ell,k_\ell}|<C_{\epsilon_0}\,e^{-\epsilon_0(\|J_\ell\|+|k_\ell|)^{1/s}},\quad \text{with } \|J_\ell\|+|k_\ell|\geq\ell.$$

Assume that $\Delta_{J_\ell k_\ell}=0$ for infinitely many values of $\ell\in\mathbb{Z}_+$. By passing to a subsequence if necessary, we may assume that $\Delta_{J_\ell k_\ell}=0$ for every $\ell\in\mathbb{Z}_+$.

Hence, if the equation $Pu=f$ has a solution u on $\mathbb{T}^{n+1}$ then, by (3 and 4), the Fourier coefficients of f must satisfy, for each $\ell\in\mathbb{Z}_+$,

(i) either $f_{J_\ell,k_\ell}=0$ or $f_{-J_\ell,-k_\ell}=0$, if $B=0$;
(ii) $[i(k_\ell+C\cdot J_\ell)-\overline{A}]f_{J_\ell,k_\ell}+B\overline{f_{-J_\ell,-k_\ell}}=0$, if $B\neq0$.

This implies that f has to satisfy infinitely many compatibility conditions. Therefore, the image $PG^s(\mathbb{T}^{n+1})$ has infinite codimension and P is not s-solvable on $\mathbb{T}^{n+1}$.

We stressed that (i) and (ii) are compatibility conditions in two different situations; that is, (i) are compatibility conditions in the linear case, while (ii) are compatibility conditions in the non $\mathbb{C}$-linear case.

Finally, assume that $\Delta_{J_\ell,k_\ell}=0$ only for a finite number of values of $\ell\in\mathbb{Z}_+$. Hence, by passing to a subsequence, we may assume

$$0<|\Delta_{J_\ell,k_\ell}|<C_{\epsilon_0}\,e^{-\epsilon_0(\|J_\ell\|+|k_\ell|)^{1/s}},\quad \ell\in\mathbb{Z}_+ \tag{7}$$

and, also, that for some m all $j_{m\ell}$'s are nonzero and have the same sign, where $j_{m\ell}$ is the m-th coordinate of J_ℓ.

Let $\Omega=\{(J_\ell,k_\ell);\,\ell\in\mathbb{Z}_+\}$ and note that Ω is an infinite set.

Assume that $B\neq0$. Let Ω_0 be an infinite subset of Ω and define

$$f(x,t)=\sum_{(J,k)\in\Omega_0}\Delta_{J,k}e^{i(J\cdot x+kt)}.$$

It follows from (7) that $f\in G^s(\mathbb{T}^{n+1})$. Note that $f_{-J,-k}=0$ for $(J,k)\in\Omega_0$, because according our assumption each $j_{m\ell}$ (the m-th coordinate of J_ℓ) must have the same sign. Let u be a solution to the equation $Pu=f$. Simple calculations show us that we can write $u=w+v$, where

$$v(x,t)=\sum_{(J,k)\in\Omega_0}Be^{-i(J\cdot x+kt)}+\sum_{(J,k)\in\Omega_0}\left[i(k+\overline{C}\cdot J)-\overline{A}\right]e^{i(J\cdot x+kt)} \tag{8}$$

and the Fourier series of w contains only frequencies $(J,k)\notin\Omega_0\cup(-\Omega_0)$. Hence, $v\in\mathscr{D}'(\mathbb{T}^{n+1})\backslash G^s(\mathbb{T}^{n+1})$ and, consequently, $u\in\mathscr{D}'(\mathbb{T}^{n+1})\backslash G^s(\mathbb{T}^{n+1})$. As before, $PG^s(\mathbb{T}^{n+1})$ has infinite codimension in $G^s(\mathbb{T}^{n+1})$.

Now, assume that $B = 0$. Then, either

$$|i(k_\ell + C \cdot J_\ell) - A| < C_{\epsilon_0}^{\frac{1}{2}} \, e^{-\frac{\epsilon_0}{2}(\|J_\ell\|+|k_\ell|)^{1/s}} \tag{9}$$

or

$$|-i(k_\ell + C \cdot J_\ell) - A| < C_{\epsilon_0}^{\frac{1}{2}} \, e^{-\frac{\epsilon_0}{2}(\|J_\ell\|+|k_\ell|)^{1/s}} \tag{10}$$

for infinitely many values of $\ell \in \mathbb{Z}_+$. By passing to a subsequence, we may assume that either (9) or (10) holds for every $\ell \in \mathbb{Z}_+$. We will assume that (9) holds for every $\ell \in \mathbb{Z}_+$ (the case (10) is analogous). Let Ω_0 be an infinite subset of Ω and define $f \in G^s(\mathbb{T}^{n+1})$ by

$$f = \sum_{(J,k)\in\Omega_0} [i(k + C \cdot J) - A] e^{i(J\cdot x + kt)},$$

Let u be a solution to the equation $Pu = f$. As before, we can write $u = w + v$, where

$$v(x,t) = \sum_{(J,k)\in\Omega_0} e^{i(J\cdot x+kt)} \in \mathscr{D}'(\mathbb{T}^{n+1})\backslash G^s(\mathbb{T}^{n+1}) \tag{11}$$

and the Fourier series of w contains only frequencies $(J,k) \notin \Omega_0 \cup (-\Omega_0)$; hence, $u \in \mathscr{D}'(\mathbb{T}^{n+1}) \setminus G^s(\mathbb{T}^{n+1})$. Therefore, P is not s-solvable on $\mathbb{T}^{n+1}$. $\square$

In the C^∞ context, we say that P (given by (2) and viewed as an operator acting in $C^\infty(\mathbb{T}^2)$) is solvable on $\mathbb{T}^{n+1}$ if for every f in a subspace of $C^\infty(\mathbb{T}^{n+1})$ of finite codimension there exists $u \in C^\infty(\mathbb{T}^{n+1})$ such that $Pu = f$ in $\mathbb{T}^{n+1}$. Also, we say that P is globally hypoelliptic if $u \in \mathscr{D}'(\mathbb{T}^{n+1})$ and $Pu \in C^\infty(\mathbb{T}^{n+1})$ imply that $u \in C^\infty(\mathbb{T}^{n+1})$.

Similiar arguments used in the proof of Theorem 1 can be used to obtain the following C^∞ version:

Theorem 2. *Let P be given by (2). Then, P is solvable on $\mathbb{T}^{n+1}$ if and only if there is a constant $\gamma > 0$ such that*

$$|\Delta_{J,k}| \geq \frac{1}{(\|J\| + |k|)^\gamma}, \quad \text{for all } (J,k) \in \mathbb{Z}^n \times \mathbb{Z} \text{ and } \|J\| + |k| \geq \gamma, \tag{12}$$

where $\Delta_{J,k}$ is given by (4).

Remark 1. Comparing Theorem 2 with Theorem 1 of [4], now $\mathsf{Im}(C) \neq 0$ is not enough to guarantee that $L = \dfrac{\partial u}{\partial t} + \sum_{j=1}^n C_j \dfrac{\partial u}{\partial x_j}$ is elliptic and, consequently, that $\Delta_{J,k}$ satisfies (12). For instance, taking $C = (i, 0, \cdots, 0)$ and $A = B = 1$ we have $\mathsf{Im}(C) \neq 0$ and $\Delta_{J,0} = 0$ for all $J = (0, j_2, \cdots, j_n)$.

The next result shows that if $|B| > |A|$ then the non $\mathbb{C}$-linearity of P is strong enough to guarantee the solvability.

Corollary 1. *Let P be given by (2). If $|B| > |A|$ then P is solvable and s-solvable on $\mathbb{T}^{n+1}$.*

Proof. We have

$$|\Delta_{J,k}| \geq |\mathsf{Re}(\Delta_{J,k})| = |-|k + C \cdot J|^2 + |A|^2 - |B|^2| \geq |B|^2 - |A|^2 > 0,$$

for all $(J, k) \in \mathbb{Z}^n \times \mathbb{Z}$. □

Corollary 2. *Let $s \geq 1$ and let $P : G^s(\mathbb{T}^2) \to G^s(\mathbb{T}^2)$ be given by*

$$Pu = \frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} - Au - B\bar{u},$$

where $A, B, C \in \mathbb{C}$. If $\mathsf{Im}C \neq 0$ then P is s-solvable on $\mathbb{T}^2$.

Proof. The vector field $L = \partial/\partial t + C\,\partial/\partial x$ is elliptic, since $\mathsf{Im}C \neq 0$. Hence, as showed in [4], $|\Delta_{j,k}|$ satisfies (12) and, therefore, (6). □

Corollary 3. *Let P be given by (2). If P is solvable on $\mathbb{T}^{n+1}$ then P is s-solvable on $\mathbb{T}^{n+1}$.*

In general the reciprocal of Corollary 3 is not true, as we can see in the example 1 below.

Conditions (6 and 12) are linked to the notion of (exponential) Liouville numbers.

Let α be an irrational number. We say that α is a *Liouville number* if for every $N \in \mathbb{Z}_+$ there is $K > 0$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < Kq^{-N} \tag{13}$$

has infinitely many solutions $p/q \in \mathbb{Q}$, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$. Also, we say that α is an *exponential Liouville number* of order $s \geq 1$ if there exists $\epsilon > 0$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < e^{-\epsilon q^{1/s}} \tag{14}$$

has infinitely many solutions $p/q \in \mathbb{Q}$, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$.

Recall that an irrational number α has an unique continued fraction expansion

$$\alpha = [a_0 : a_1, a_2, a_3, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{Z}_+$ for $n \geq 1$.

Example 1. Let $s \geq 1$ and let $P : G^s(\mathbb{T}^2) \to G^s(\mathbb{T}^2)$ be given by

$$Pu = \frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} - iu - \bar{u},$$

where $\alpha = [1 : 10, 10^{2!}, \cdots, 10^{n!}, \cdots]$.

The irrational α is a Liouville number (see [15]), but it is not an exponential Liouville number of order s, for any $s \geq 1$ (see [1] and [13]).

We claim that P is not solvable on $\mathbb{T}^2$ (viewed as an operator acting in $C^\infty(\mathbb{T}^2)$), but P is s-solvable. Indeed, we have that

$$|\Delta_{j,k}| = |k - \alpha j|^2.$$

Since α is a Liouville number, for every $\gamma > 0$ there is a sequence $(p_\ell, q_\ell) \in \mathbb{Z}^2$, with $q_\ell \to \infty$, such that $|q_\ell \alpha - p_\ell| < q_\ell^{-\gamma}$, for all $\ell \geq 1$; equivalently, for every $\gamma > 0$ there is a sequence $(p_\ell, q_\ell) \in \mathbb{Z}^2$, with $q_\ell \to \infty$, such that $|q_\ell \alpha - p_\ell| < (|p_\ell| + q_\ell)^{-\gamma}$, for all $\ell \geq 1$. Hence, (12) is not satisfied and, consequently, P is not solvable on $\mathbb{T}^2$. On the other hand, since α is not an exponential Liouville number of order s, for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $|j\alpha - k| \geq C_\epsilon e^{-\epsilon(|k|+|j|)^{1/s}}$, for all $(j,k) \in (\mathbb{Z} \times \mathbb{Z}) \setminus \{(0,0)\}$. Hence, (6) is satisfied and, consequently, P is s-solvable on $\mathbb{T}^2$. $\square$

The arguments used in the proof of Theorem 1 can be used to prove the following.

Theorem 3. *Let P be given by (2). Then, P is s-globally hypoelliptic if and only if (6) is satisfied. Also, P is globally hypoelliptic if and only if (12) is satisfied.*

It follows from Theorems 1, 2, and 3 that

Corollary 4. *Let P be given by (2). Then, P is s-solvable on $\mathbb{T}^{n+1}$ if and only if P is s-globally hypoelliptic. Also, P is solvable on $\mathbb{T}^{n+1}$ if and only if P is globally hypoelliptic.*

Now, let us return to our problem in a special situation: the case where

$$L = \frac{\partial u}{\partial t} + \sum_{j=1}^{n} C_j \frac{\partial u}{\partial x_j}$$

is a real vector field; that is, $C_j \in \mathbb{R}$, for $j = 1, \cdots, n$.

Natural Diophantine conditions $(\mathrm{DC})_s$ and (DC) appear. For fixed $s \geq 1$, we say that $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$ satisfies $(\mathrm{DC})_s$ if for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$(\mathrm{DC})_s \qquad |k + \xi \cdot J - \eta| \geq C_\epsilon e^{-\epsilon(\|J\|+|k|)^{1/s}},$$

for all $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$, with $\|J\| + |k| \geq C_\epsilon$. Also, we say that $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$ satisfies (DC) if there is a constant $\gamma > 0$ such that

$$(\mathrm{DC}) \qquad |k + \xi \cdot J - \eta| \geq \frac{1}{(\|J\| + |k|)^\gamma}$$

for all $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$ and $\|J\| + |k| \geq \gamma$.

In the situation where $\mathsf{Im}(C) = 0$, Theorems 1 and 3 can be rewritten as follows:

Theorem 4. *Let $s \geq 1$ and let P be given by (2). Assume that $\mathsf{Im}(C) = 0$. Then, P is s-solvable on $\mathbb{T}^{n+1}$ if and only if one of the following conditions is satisfied:*

(i) $|B| > |A|$;
(ii) $|B| < |A|$, *and* $\mathsf{Re}(A) \neq 0$;
(iii) *the vector* $(C, \sqrt{|A|^2 - |B|^2})$ *is in* $\mathbb{R}^{n+1}$, *and satisfies* $(DC)_s$.

Also, we have:

Theorem 5. *Let $s \geq 1$ and let P be given by (2). Assume that $\mathsf{Im}(C) = 0$. Then, P is not s-globally hypoelliptic if and only if the following condition is satisfied:*

(iv) $|B| \leq |A|$, the vector $(C, \sqrt{|A|^2 - |B|^2})$ does not satisfy $(DC)_s$ and, moreover, one has $\mathsf{Re}(A) = 0$ when $|B| < |A|$.

The proof of Theorems 4 and 5 can be obtained by applying the similar arguments that in the proof of Theorems 1 and 2 of [4] and it will be omitted here.

Also, we could state C^∞ versions of the Theorems 4 and 5 replacing $(\mathrm{DC})_s$ by (DC).

3. A Class of Operators with Variable Coefficients

In this section we will consider a class of operators $P : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$ with variable coefficients.

Let $s \geq 1$ and let

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} (p_j(t) + i\lambda_j q_j(t)) \frac{\partial}{\partial x_j}, \tag{15}$$

be a complex vector field defined on $\mathbb{T}^n_x \times \mathbb{T}^1_t$, where $q, p_j \in G^s(\mathbb{T}^1_t; \mathbb{R})$, $j = 1, \cdots, n$, $q_j \not\equiv 0$ for some j, and $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$.

We will assume that L satisfies the *Nirenberg-Treves* condition $(\mathscr{P})$; hence, L is locally solvable (see, for instance, [5] or [19]; also, [10]).

Our assumption implies that L has the form

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} (p_j(t) + i\lambda_j q(t)) \frac{\partial}{\partial x_j}, \quad q \not\equiv 0, \tag{16}$$

where $q \in G^s(\mathbb{T}^1)$ does not change sign on $\mathbb{T}^1$ (see [9]; also, [6])). There is no loss of generality in assuming that $q(t) \geq 0$ for all $t \in \mathbb{T}^1$.

Let

$$p_{0j} = \frac{1}{2\pi} \int_0^{2\pi} p_j(t) dt\,, \quad m_j(t) = \int_0^t (p_j(\tau) - p_{0j}) d\tau\,,$$

and

$$m(t) = (m_1(t), \cdots, m_n(t)).$$

By using partial Fourier series in the variables $(x_1, \ldots, x_n)$ we define the operator $T : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$ given by

$$Tu(x,t) = \sum_{J \in \mathbb{Z}^n} \widehat{Tu}(J,t) e^{ix\cdot J}, \tag{17}$$

with

$$\widehat{Tu}(J,t) = \hat{u}(J,t)\, e^{i\int_0^t m(\tau)\cdot J d\tau}, \quad \text{for all} \quad J \in \mathbb{Z}^n.$$

As showed in [8], T is well-defined and

$$TLT^{-1} = \frac{\partial}{\partial t} - \sum_{j=1}^{n} (p_{0j} + i\lambda_j q(t)) \frac{\partial}{\partial x_j},$$

where T^{-1}, the inverse of T, is given by

$$\widehat{T^{-1}v}(J,t) = \hat{v}(J,t)\, e^{-i\int_0^t m(\tau)\cdot J d\tau}, \quad \text{for all} \quad J \in \mathbb{Z}^n.$$

From now on we will assume that our operator L has the form

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} (p_{0j} + i\lambda_j q(t)) \frac{\partial}{\partial x_j}. \tag{18}$$

3.1. Complex Case

Assume that $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \setminus \{0\}$.

Let

$$P : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$$

be the operator defined by

$$Pu = Lu - (r(t) + i\delta q(t))u - \alpha q(t)\bar{u}, \tag{19}$$

where L is given by (18), $r \in G^s(\mathbb{T}^1; \mathbb{R})$, $\delta \in \mathbb{R}$ and $\alpha \in \mathbb{C} \setminus \{0\}$.

Let

$$p_0 = (p_{01}, \cdots, p_{0n}) \quad \text{and} \quad \lambda = (\lambda_1, \cdots, \lambda_n),$$

where p_{0j} and λ_j are given in (18). Define

$$\mathscr{Q}(t) = \int_0^t q(\sigma) d\sigma\,, \ \ \mathscr{R}(t) = \int_0^t r(\sigma) d\sigma\,, \ \ q_0 = \int_0^{2\pi} q(\sigma) d\sigma\,, \text{ and } r_0 = \int_0^{2\pi} r(\sigma) d\sigma\,.$$

We have $q_0 > 0$, since $0 \leq q \not\equiv 0$.

Let

$$A_0 = r_0 + i\delta q_0, \quad B_0 = \alpha q_0, \quad \text{and} \quad C_0 = 2\pi p_0 + i q_0 \lambda.$$

Note that $C_0 = (2\pi p_{01} + i\lambda_1 q_0, \cdots, 2\pi p_{0n} + i\lambda_n q_0) \in \mathbb{C}^n$. Also, for each $J \in \mathbb{Z}^n$ let

$$\rho_J = \sqrt{(\lambda \cdot J - i\delta)^2 + |\alpha|^2},$$

where we choose ρ_J to have $\mathsf{Re}(\rho_J) \geq 0$.

Now, we are ready to present the main result of this section.

Theorem 6. *Let P be given by (19). Assume that the coefficients of P satisfy:*

(I) $|\alpha| \neq |\delta|$;

(II) there is no $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$ satisfying

$$\begin{cases} \mathsf{Re}(A_0(2k\pi + J\cdot \overline{C_0})) = 0 \\ \quad |2k\pi + J\cdot C_0|^2 = |A_0|^2 - |B_0|^2 \end{cases} ; \text{ and}$$

(III) for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$\min\left\{\left|e^{-\rho_J q_0} - e^{r_0 + 2\pi i J\cdot p_0}\right|,\ \left|1 - e^{r_0 - \rho_J q_0 + 2\pi i J\cdot p_0}\right|\right\} \geq C_\epsilon e^{-\epsilon \|J\|^{1/s}}$$

for all $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$, with $\|J\| \geq C_\epsilon$.

Then, for every $f \in G^s(\mathbb{T}^{n+1})$, there is $u \in G^s(\mathbb{T}^{n+1})$ solution to the equation $Pu = f$ in $\mathbb{T}^{n+1}$.

Proof. The proof of this Theorem, which amounts to an adaptation of the proof of Theorem 7 of [4], will be sketched here.

Given $f \in G^s(\mathbb{T}^{n+1})$ it will be found $u \in G^s(\mathbb{T}^{n+1})$ solution to $Pu = f$ in $\mathbb{T}^{n+1}$. By using partial Fourier series in the variables $(x_1, \cdots, x_n)$ we have

$$u(x,t) = \sum_{J\in\mathbb{Z}^n} u_J(t) e^{iJ\cdot x} \quad \text{and}$$
$$f(x,t) = \sum_{J\in\mathbb{Z}^n} f_J(t) e^{iJ\cdot x}.$$

Hence, for each $J \in \mathbb{Z}^n$ the equation $Pu = f$ leads to

$$\begin{cases} u_J' - [i(p_0 + i\lambda q(t))\cdot J + r(t) + i\delta q(t)]u_J - \alpha q(t)\overline{u_{-J}} = f_J \\ \overline{u_{-J}}' - [i(p_0 - i\lambda q(t))\cdot J + r(t) - i\delta q(t)]\overline{u_{-J}} - \overline{\alpha} q(t) u_J = \overline{f_{-J}} \end{cases} \tag{20}$$

For each $J \in \mathbb{Z}^n$, let

$$w_J = \begin{pmatrix} u_J \\ \overline{u_{-J}} \end{pmatrix} \quad \text{and} \quad \mathbb{F}_J = \begin{pmatrix} f_J \\ \overline{f_{-J}} \end{pmatrix}.$$

We can rewrite (20) as

$$w_J' = M_J w_J + \mathbb{F}_J, \tag{21}$$

where

$$M_J = \begin{pmatrix} i(p_0 + i\lambda q(t))\cdot J + r(t) + i\delta q(t) & \alpha q(t) \\ \overline{\alpha} q(t) & i(p_0 - i\lambda q(t))\cdot J + r(t) - i\delta q(t) \end{pmatrix}.$$

In order for the function w_J to be a 2π-periodic solution of (21), the function

$$y_J = e^{-iJ\cdot p_0 t - \mathscr{R}(t)} w_J \tag{22}$$

has to satisfy

$$y_J' = q(t) N_J y_J + e^{-iJ\cdot p_0 t - \mathscr{R}(t)} \mathbb{F}_J, \tag{23}$$

and, also,

$$y_J(0) = e^{2\pi i J\cdot p_0 + r_0} y_J(2\pi)\,,$$

where

$$N_J = \begin{pmatrix} -\lambda\cdot J + i\delta & \alpha \\ \overline{\alpha} & \lambda\cdot J - i\delta \end{pmatrix}.$$

The eigenvalues ρ_J and σ_J of N_J are given by

$$\rho_J = \sqrt{(\lambda\cdot J - i\delta)^2 + |\alpha|^2)} \quad \text{and} \quad \sigma_J = -\rho_J;$$

recall that $\mathsf{Re}(\rho_J) \geq 0$. It follows from (I) that $\rho_J \neq 0$ and, consequently, N_J is invertible. It is easy to see that the eigenvectors of N_J corresponding to $\pm\rho_J$ are

$$V_J^{\pm} = \begin{pmatrix} \alpha \\ (\lambda\cdot J - i\delta) \pm \rho_J \end{pmatrix}.$$

For each $J \in \mathbb{Z}^n$, let $T_J = (V_J^+ \;\; V_J^-)$. Then

$$T_J^{-1} N_J T_J = \rho_J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and (23) becomes

$$y_J' = q(t)\rho_J T_J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_J^{-1} y_J + e^{-iJ\cdot p_0 t - \mathscr{R}(t)} T_J T_J^{-1} \mathbb{F}_J. \tag{24}$$

For each $J \in \mathbb{Z}^n$, let $z_J = T_J^{-1} y_J$. The differential equation (24) leads us to

$$z_J' = \rho_J q(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z_J + e^{-iJ\cdot p_0 t - \mathscr{R}(t)} T_J^{-1} \mathbb{F}_J \tag{25}$$

restricted to $z_J(0) = e^{2\pi i J\cdot p_0 + r_0} z_J(2\pi)$.

For $J \in \mathbb{Z}^n$, let

$$\mathsf{Z}_J(t) = \begin{pmatrix} e^{\rho_J \tilde{\mathscr{Q}}(t)} & 0 \\ 0 & e^{-\rho_J \mathscr{Q}(t)} \end{pmatrix},$$

where $\tilde{\mathscr{Q}}(t) = -\displaystyle\int_t^{2\pi} q(\sigma)d\sigma$.

We will seek for solutions of (25) in the form $z_J(t) = \mathsf{Z}_J(t) C_J(t)$. Let

$$G_J(t) = \begin{pmatrix} G_{1J}(t) \\ G_{2J}(t) \end{pmatrix} = T_J^{-1} \mathbb{F}_J(t).$$

The equation (25) leads us to

$$\mathsf{Z}_J(t) C_J'(t) = e^{-iJ\cdot p_0 t - \mathscr{R}(t)} G_J(t).$$

Hence,

$$C_J(t) = \begin{pmatrix} -\displaystyle\int_t^{2\pi} e^{-\rho_J \tilde{\mathscr{Q}}(\sigma)} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{1J}(\sigma)d\sigma \\ \displaystyle\int_0^t e^{\rho_J \mathscr{Q}(\sigma)} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{2J}(\sigma)d\sigma \end{pmatrix} + \mathbb{K}_J,$$

for some $\mathbb{K}_J \in \mathbb{R}^2$; consequently,

$$z_J(t) = \begin{pmatrix} -\displaystyle\int_t^{2\pi} e^{\rho_J (\tilde{\mathscr{Q}}(t) - \tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{1J}(\sigma)d\sigma \\ \displaystyle\int_0^t e^{\rho_J (\mathscr{Q}(\sigma) - \mathscr{Q}(t))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{2J}(\sigma)d\sigma \end{pmatrix} + \mathsf{Z}_J(t)\mathbb{K}_J.$$

Since z_J has to satisfy $z_J(0) = e^{iJ\cdot p_0 2\pi + r_0} z_J(2\pi)$, we should have

$$\begin{aligned} &\mathsf{Z}_J(0)\mathbb{K}_J + \begin{pmatrix} -\displaystyle\int_0^{2\pi} e^{\rho_J (\tilde{\mathscr{Q}}(0) - \tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{1J}(\sigma)d\sigma \\ 0 \end{pmatrix} \\ &\quad = e^{iJ\cdot p_0 2\pi + r_0} \begin{pmatrix} 0 \\ \displaystyle\int_0^{2\pi} e^{\rho_J (\mathscr{Q}(\sigma) - \mathscr{Q}(2\pi))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)} G_{2J}(\sigma)d\sigma \end{pmatrix} \\ &\quad + e^{2\pi iJ\cdot p_0 + r_0} \mathsf{Z}_J(2\pi)\mathbb{K}_J \,. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left(\mathsf{Z}_J(0) - e^{2\pi iJ\cdot p_0 + r_0}\mathsf{Z}_J(2\pi)\right)\mathbb{K}_J \\ &= \begin{pmatrix} \displaystyle\int_0^{2\pi} e^{\rho_J (\tilde{\mathscr{Q}}(0) - \tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)}\, G_{1J}(\sigma)d\sigma \\ \displaystyle\int_0^{2\pi} e^{\rho_J (\mathscr{Q}(\sigma) - \mathscr{Q}(2\pi))} e^{-iJ\cdot p_0(\sigma - 2\pi) - (\mathscr{R}(\sigma) - r_0)}\, G_{2J}(\sigma)d\sigma \end{pmatrix}. \end{aligned} \tag{26}$$

Simple calculations show that condition (II) implies that the matrix

$$\mathsf{Z}_J(0) - e^{2\pi iJ\cdot p_0 + r_0}\mathsf{Z}_J(2\pi) = \begin{pmatrix} e^{-\rho_J q_0} - e^{r_0 + 2\pi iJ\cdot p_0} & 0 \\ 0 & 1 - e^{-\rho_J q_0 + r_0 + 2\pi iJ\cdot p_0} \end{pmatrix}$$

is invertible. Then,

$$\mathbb{K}_J = \begin{pmatrix} \displaystyle\int_0^{2\pi} \frac{e^{\rho_J (\tilde{\mathscr{Q}}(0) - \tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma - \mathscr{R}(\sigma)}}{e^{-\rho_J q_0} - e^{2\pi iJ\cdot p_0 + r_0}}\, G_{1J}(\sigma)d\sigma \\ \displaystyle\int_0^{2\pi} \frac{e^{\rho_J (\mathscr{Q}(\sigma) - \mathscr{Q}(2\pi))} e^{-iJ\cdot p_0(\sigma - 2\pi) - (\mathscr{R}(\sigma) - r_0)}}{1 - e^{2\pi iJ\cdot p_0 + r_0 - \rho_J q_0}}\, G_{2J}(\sigma)d\sigma \end{pmatrix}$$

and, therefore,

$$z_J(t) = \begin{pmatrix} -\int_t^{2\pi} e^{\rho_J(\tilde{\mathscr{Q}}(t)-\tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma-\mathscr{R}(\sigma)}\, G_{1J}(\sigma)d\sigma \\ \int_0^t e^{\rho_J(\mathscr{Q}(\sigma)-\mathscr{Q}(t))} e^{-iJ\cdot p_0\sigma-\mathscr{R}(\sigma)}\, G_{2J}(\sigma)d\sigma \end{pmatrix}$$
$$+ \begin{pmatrix} e^{\rho_J\tilde{\mathscr{Q}}(t)} \int_0^{2\pi} \dfrac{e^{\rho_J(\tilde{\mathscr{Q}}(0)-\tilde{\mathscr{Q}}(\sigma))} e^{-iJ\cdot p_0\sigma-\mathscr{R}(\sigma)}}{e^{-\rho_J q_0} - e^{2\pi iJ\cdot p_0+r_0}}\, G_{1J}(\sigma)d\sigma \\ e^{-\rho_J\mathscr{Q}(t)} \int_0^{2\pi} \dfrac{e^{\rho_J(\mathscr{Q}(\sigma)-\mathscr{Q}(2\pi))} e^{-iJp_0\sigma-\mathscr{R}(\sigma)}}{1-e^{2\pi iJ\cdot p_0+r_0-\rho_J q_0}}\, G_{2J}(\sigma)d\sigma \end{pmatrix}.$$

Finally, since $w_J = e^{iJ\cdot p_0 t+\mathscr{R}(t)} y_J$ and $y_J = T_J z_J$ we have

$$u_J(t) = \alpha e^{iJ\cdot p_0 t+\mathscr{R}(t)} (z_{1J}(t) + z_{2J}(t)),$$

where z_{1J} and z_{2J} are the components of z_J.

It follows at once from (III) that z_{1J} and z_{2J} decay rapidly. Analogous estimates can be obtained for the derivatives $z_{1J}^{(m)}$ and $z_{2J}^{(m)}$, $m \in \mathbb{N}$. Hence, (u_J) is of rapid decay as $\|J\| \to \infty$.

Therefore, the sequence (u_J) defines a G^s function $u(x,t) = \sum_{J\in\mathbb{Z}^n} u_J(t)e^{iJ\cdot x}$ solution to $Pu = f$ in $\mathbb{T}^{n+1}$. □

Similar arguments can be used to prove the following version, where the coefficients of P can be taken only in $C^\infty(\mathbb{T}^{n+1})$ instead $G^s(\mathbb{T}^{n+1})$:

Theorem 7. *Let P be given by (19). Assume that the coefficients of P satisfy:*

(I) $|\alpha| \neq |\delta|$;
(II) there is no $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$ *satisfying*

$$\begin{cases} \mathsf{Re}(A_0(2k\pi + J\cdot\overline{C_0})) = 0 \\ |2k\pi + J\cdot C_0|^2 = |A_0|^2 - |B_0|^2 \end{cases}; \text{ and}$$

(III) there is $\gamma > 0$ *such that*

$$\min\left\{\left|e^{-\rho_J q_0} - e^{r_0+2\pi iJ\cdot p_0}\right|,\ \left|1-e^{r_0-\rho_J q_0+2\pi iJ\cdot p_0}\right|\right\} \geq \frac{1}{|J|^\gamma}$$

for all $(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$, *with* $|J| \geq \gamma$.

Then, for every $f \in C^\infty(\mathbb{T}^{n+1})$, *there is* $u \in C^\infty(\mathbb{T}^{n+1})$ *solution to the equation* $Pu = f$ *in* $\mathbb{T}^{n+1}$.

Comparing Theorem 7 with Theorem 7 of [4], note that conditions (I) and (II) do not imply (III) when we are working on $\mathbb{T}^{n+1}$, with $n \geq 2$. For instance:

Example 2. Consider $P : C^\infty(\mathbb{T}^3) \to C^\infty(\mathbb{T}^3)$ defined by

$$Pu = \frac{\partial u}{\partial t} - i\cos^2(t)\sum_{j=1}^{2}\frac{\partial}{\partial x_j} - i\sqrt{2}\cos^2(t)u - \cos^2(t)\bar{u}.$$

By using the notation of Theorem 7 we have: $\alpha = 1$, $\delta = \sqrt{2}$, $A_0 = -i\sqrt{2}\pi$, $B_0 = \pi$, $C_0 = i(\pi,\pi)$, $\lambda = (1,1)$, $r_0 = 0$, $p_0 = (0,0)$, and $q_0 = \pi$.

Note that $|\alpha| = 1 < \sqrt{2} = |\delta|$; hence, (I) is satisfied.

Also, for $J = (j_1, j_2)$, we have:

$$\mathsf{Re}(A_0(2k\pi + J\cdot\overline{C_0})) = \mathsf{Re}(-i\sqrt{2}\pi(2k\pi - i(j_1,j_2)\cdot(\pi,\pi)) = -\sqrt{2}\pi^2(j_1+j_2);$$

hence,

$$\mathsf{Re}(A_0(2k\pi + J\cdot\overline{C_0})) = 0 \;\Rightarrow\; j_2 = -j_1.$$

On the other hand, for $J = (j,-j)$ we have

$$|2k\pi + J\cdot C_0|^2 = |2k\pi + i(j,-j)\cdot(\pi,\pi)|^2 = (2k)^2\pi^2 \neq \pi^2 = |A_0|^2 - |B_0|^2,$$

for all $k \in \mathbb{Z}$. Hence, (II) is satisfied.

Finally, for $J = (j,-j)$ we have $\rho_J = i$; hence,

$$1 - e^{r_0 - \rho_J q_0 + 2\pi i J\cdot p_0} = 1 - e^{-i\pi} = 0$$

and, therefore, (III) is not satisfied.

3.2. Real Case

Assume $\lambda = 0$ in (18). In this case L is a real vector field in the form

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} p_{0j}\frac{\partial}{\partial x_j}, \tag{27}$$

where $p_{0j} \in \mathbb{R}$, for all $j = 1, \cdots, n$.

We define the operator

$$P : G^s(\mathbb{T}^{n+1}) \to G^s(\mathbb{T}^{n+1})$$

given by

$$Pu = Lu - (r(t) + i\delta q(t))u - \alpha q(t)\bar{u}, \quad q \not\equiv 0, \tag{28}$$

where L is given by (27), $q, r \in G^s(\mathbb{T}^1;\mathbb{R})$, $q(t) \geq 0$ for all $t \in \mathbb{T}^1$, $\delta \in \mathbb{R}$ and $\alpha \in \mathbb{C}\setminus\{0\}$.

Theorem 8. *Let P be given by (28). Using the same notation of theorem 7, assume that one of the following conditions is satisfied:*

(i) $|B_0| > |A_0|$;
(ii) $|B_0| < |A_0|$, $|\alpha| > |\delta|$ *and*
(⋆) $\nexists\,(J,k) \in \mathbb{Z}^n \times \mathbb{Z}$ *solution for* $\begin{cases} \mathsf{Re}(A_0(k + J\cdot p_0)) = 0 \\ 4\pi^2|k + J\cdot p_0|^2 = |A_0|^2 - |B_0|^2 \end{cases}$;
(iii) $|\alpha| < |\delta|$ *and* $r_0 \neq 0$;
(iv) $|\alpha| < |\delta|$, $r_0 = 0$, *(⋆)*

holds and, also, the following Diophantine condition holds:

$(DC)'_s$ *for every* $\epsilon > 0$ *there is* $C_\epsilon > 0$ *such that*

$$|2k\pi + 2\pi J \cdot p_0 - q_0\sqrt{\delta^2 - |\alpha|^2}| \geq C_\epsilon e^{-\epsilon\|J\|^{1/s}}$$

for all $(J, k) \in \mathbb{Z}^n \times \mathbb{Z}$, *with* $\|J\| \geq C_\epsilon$.

Then, given $f \in G^s(\mathbb{T}^{n+1})$ *there is a solution* $u \in G^s(\mathbb{T}^{n+1})$ *of the equation* $Pu = f$ *in* $\mathbb{T}^{n+1}$.

We stressed that the Diophantine condition $(DC)'_s$ is slightly stronger than $(DC)_s$ given in Theorem 4.

The proof of Theorem 8 is obtained by proceeding as in the proof of Theorem 7 and by using the following:

Lemma 1. *The diophantine condition* $(DC)'_s$ *is equivalent to*

$(DC)''_s$ *for every* $\epsilon > 0$ *there is* $C_\epsilon > 0$ *such that*

$$|e^{i(2\pi J \cdot p_0 - q_0\sqrt{\delta^2 - |\alpha|^2})} - 1| \geq C_\epsilon e^{-\epsilon\|J\|^{1/s}}$$

for all $(J, k) \in \mathbb{Z}^n \times \mathbb{Z}$, *with* $\|J\| \geq C_\epsilon$.

The proof of Lemma 1 is a simple adaptation of that of Lemma 13 in [4] and we will omit here.

As done for Theorem 6, it is possible state a C^∞ version of Theorem 8.

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