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Hypersurfaces of two space forms and conformally flat hypersurfaces

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Abstract We address the problem of determining the hypersurfaces $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ with dimension $n \geq 3$ of a pseudo-Riemannian space form of dimension $n+1$, constant curvature c and index $s \in \{0, 1\}$ for which there exists another isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$. For $n \geq 4$, we provide a complete solution by extending results for $s = 0 = \tilde{s}$ by do Carmo and Dajczer (Proc Am Math Soc 86:115–119, 1982) and by Dajczer and Tojeiro (J Differ Geom 36:1–18, 1992). Our main results are for the most interesting case $n = 3$, and these are new even in the Riemannian case $s = 0 = \tilde{s}$. In particular, we characterize the solutions that have dimension $n = 3$ and three distinct principal curvatures. We show that these are closely related to conformally flat hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures, and we obtain a similar characterization of the latter that improves a theorem by Hertrich-Jeromin (Beitr Algebra Geom 35:315–331, 1994).

Keywords Hypersurfaces of two space forms · Conformally flat hypersurfaces · Holonomic hypersurfaces

Mathematics Subject Classification 53B25

We denote by $\mathbb{Q}_s^N(c)$ a pseudo-Riemannian space form of dimension N, constant sectional curvature c and index $s \in \{0, 1\}$, that is, $\mathbb{Q}_s^N(c)$ is either a Riemannian or Lorentzian space form of constant curvature c, corresponding to $s = 0$ or $s = 1$, respectively. By a hypersurface

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$f: M^n \to \mathbb{Q}_s^{n+1}(c)$ we always mean an isometric immersion of a *Riemannian* manifold M^n of dimension n into $\mathbb{Q}_s^{n+1}(c)$, thus f is a *space-like* hypersurface if $s = 1$.

One of the main purposes of this paper is to address the following

Problem $*$: For which hypersurfaces $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ of dimension $n \geq 3$ does there exist another isometric immersion $\tilde{f}: M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$?

This problem was studied for $s = 0 = \tilde{s}$ and $n \geq 4$ by do Carmo and Dajczer in [3], and by Dajczer and the second author in [5]. Some partial results in the most interesting case $n = 3$ were also obtained in [5]. Including Lorentzian ambient space forms in our study of Problem $*$ was motivated by our investigation in [1] of submanifolds of codimension two and constant curvature $c \in (0, 1)$ of $\mathbb{S}^5 \times \mathbb{R}$, which turned out to be related to hypersurfaces $f: M^3 \to \mathbb{S}^4$ for which M^3 also admits an isometric immersion into the Lorentz space $\mathbb{R}_1^4 = \mathbb{Q}_1^4(0)$.

We first state our results for the case $n \geq 4$. The next one extends a theorem due to do Carmo and Dajczer [3] in the case $s = 0 = \tilde{s}$. Here and in the sequel, for $s \in \{0, 1\}$ we denote $\epsilon_s = -2s + 1$.

Proposition 1 *Let $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ be a hypersurface of dimension $n \geq 4$. If there exists another isometric immersion $\tilde{f}: M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$, then $c < \tilde{c}$ if $s = 0$ and $\tilde{s} = 1$ (respectively, $c > \tilde{c}$ if $s = 1$ and $\tilde{s} = 0$) and f has a principal curvature λ of multiplicity at least $n - 1$ everywhere satisfying $\rho := \epsilon_{\tilde{s}}(c - \tilde{c} + \epsilon_s \lambda^2) \geq 0$. Moreover, at any $x \in M^n$ the following holds:*

- (i) *if $\lambda = 0$ or f is umbilical with $\rho > 0$, then $\tilde{f}$ is umbilical;*
- (ii) *if f is umbilical and $\rho = 0$, then 0 is a principal curvature of $\tilde{f}$ with multiplicity at least $n - 1$;*
- (iii) *if $\lambda \neq 0$ with multiplicity $n - 1$, then $\tilde{f}$ has a principal curvature $\tilde{\lambda}$, with $\tilde{\lambda}^2 = \rho$, which has the same eigenspace as λ.*

Thus, Problem $*$ has no solutions if $n \geq 4$ and either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$ or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, while, in the remaining cases, having a principal curvature λ of multiplicity at least $n - 1$ satisfying $\epsilon_{\tilde{s}}(c - \tilde{c} + \epsilon_s \lambda^2) \geq 0$ is a necessary condition for a solution. In those cases, having a principal curvature of *constant* multiplicity n or $n - 1$ satisfying the preceding condition is also sufficient for simply connected hypersurfaces.

Proposition 2 *Let $f: M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$ be an isometric immersion of a simply connected Riemannian manifold. Given $\tilde{c} \neq c$ and $\tilde{s} \in \{0, 1\}$, assume that $c < \tilde{c}$ if $s = 0$ and $\tilde{s} = 1$, and that $c > \tilde{c}$ if $s = 1$ and $\tilde{s} = 0$. If f has a principal curvature λ of (constant) multiplicity either $n - 1$ or n satisfying $\rho := \epsilon_{\tilde{s}}(c - \tilde{c} + \epsilon_s \lambda^2) \geq 0$, then M^n admits an isometric immersion into $\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$, which is unique up to congruence if $\rho > 0$.*

The next result, proved by Dajczer and the second author in [5] when $s = 0 = \tilde{s}$, shows how any solution $f: M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$, of Problem $*$ arises.

Proposition 3 *Let $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ and $\tilde{f}: M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$, $n \geq 4$, be isometric immersions with, say, $c > \tilde{c}$. If $s = 0$, assume that $\tilde{s} = 0$. Then, for $s = \tilde{s}$ (respectively, $s = 1$ and $\tilde{s} = 0$), there exist, locally on an open dense subset of M^n, isometric embeddings*

$$H: \mathbb{Q}_s^{n+1}(\tilde{c}) \to \mathbb{Q}_s^{n+2}(\tilde{c}) \quad \textit{and} \quad i: \mathbb{Q}_s^{n+1}(c) \to \mathbb{Q}_s^{n+2}(\tilde{c})$$

(respectively, $H: \mathbb{Q}_s^{n+1}(c) \to \mathbb{Q}_s^{n+2}(c)$ and $i: \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c}) \to \mathbb{Q}_s^{n+2}(c)$), with i umbilical, and an isometry

$$\Psi: \bar{M}^n := H(\mathbb{Q}_s^{n+1}(\tilde{c})) \cap i(\mathbb{Q}_s^{n+1}(c)) \to M^n$$

(respectively, $\Psi: \bar{M}^n := H(\mathbb{Q}_s^{n+1}(c)) \cap i(\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})) \to M^n$) such that

$$f \circ \Psi = i^{-1}|_{\bar{M}^n} \quad and \quad \tilde{f} \circ \Psi = H^{-1}|_{\bar{M}^n}.$$

(respectively, $f \circ \Psi = H^{-1}|_{\bar{M}^n}$ *and* $\tilde{f} \circ \Psi = i^{-1}|_{\bar{M}^n}$*).*

Proposition 3 explains the existence of a principal curvature λ of multiplicity at least $n-1$ for a solution $f: M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$, of Problem $*$: the (images by f of the) leaves of the distribution on M^n given by the eigenspaces of λ are the intersections with $i(\mathbb{Q}_s^{n+1}(\tilde{c}))$ of the (images by H of the) relative nullity leaves of H, which have dimension at least n.

Next we consider Problem $*$ for hypersurfaces of dimension $n = 3$. The following result provides the solutions in two ("dual") special cases.

Theorem 4 *Let* $f: M^3 \to \mathbb{Q}_s^4(c)$ *be a hypersurface for which there exists an isometric immersion* $\tilde{f}: M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ *with* $\tilde{c} \neq c$.

- (a) *Assume that* f *has a principal curvature of multiplicity two. If either* $c > \tilde{c}$, $s = 0$ *and* $\tilde{s} = 1$, *or if* $c < \tilde{c}$, $s = 1$ *and* $\tilde{s} = 0$, *then* f *is a rotation hypersurface whose profile curve is a* $\tilde{c}$*-helix in a totally geodesic surface* $\mathbb{Q}_s^2(c)$ *of* $\mathbb{Q}_s^4(c)$ *and* $\tilde{f}$ *is a generalized cone over a surface with constant curvature in an umbilical hypersurface* $\mathbb{Q}_{\tilde{s}}^3(\bar{c})$ *of* $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, $\bar{c} \geq \tilde{c}$. *Otherwise, either the same conclusion holds or* f *and* $\tilde{f}$ *are locally given on an open dense subset as described in Proposition 3.*
- (b) *If one of the principal curvatures of* f *is zero, then* f *is a generalized cone over a surface with constant curvature in an umbilical hypersurface* $\mathbb{Q}_s^3(\bar{c})$ *of* $\mathbb{Q}_s^4(c)$, $\bar{c} \geq c$, *and* $\tilde{f}$ *is a rotation hypersurface whose profile curve is a* c*-helix in a totally geodesic surface* $\mathbb{Q}_{\tilde{s}}^2(\tilde{c})$ *of* $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$.

By a *generalized cone* over a surface $g: M^2 \to \mathbb{Q}_s^3(\bar{c})$ in an umbilical hypersurface $\mathbb{Q}_s^3(\bar{c})$ of $\mathbb{Q}_s^4(c)$, $\bar{c} \geq c$, we mean the hypersurface parametrized by (the restriction to the subset of regular points of) the map $G: M^2 \times \mathbb{R} \to \mathbb{Q}_s^4(c)$ given by

$$G(x,t) = \exp_{g(x)}(t\xi(g(x))),$$

where ξ is a unit normal vector field to the inclusion $i: \mathbb{Q}_s^3(\bar{c}) \to \mathbb{Q}_s^4(c)$ and exp is the exponential map of $\mathbb{Q}_s^4(c)$. A *c-helix* in $\mathbb{Q}_s^2(\tilde{c}) \subset \mathbb{R}_{s+\epsilon_0}^3$ with respect to a unit vector $v \in \mathbb{R}_{s+\epsilon_0}^3$ is a unit-speed curve $\gamma: I \to \mathbb{Q}_s^2(\tilde{c}) \subset \mathbb{R}_{s+\epsilon_0}^3$ such that the height function $\gamma_v = \langle \gamma, v \rangle$ satisfies $\gamma_v'' + c\gamma_v = 0$. Here $\epsilon_0 = 0$ or 1, corresponding to $\tilde{c} > 0$ or $\tilde{c} < 0$, respectively.

In order to deal with the generic case of Problem $*$ for hypersurfaces of dimension 3, we need to recall the notion of holonomic hypersurfaces. We call a hypersurface $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ *holonomic* if M^n carries global orthogonal coordinates $(u_1, \ldots, u_n)$ such that the coordinate vector fields $\partial_j = \dfrac{\partial}{\partial u_j}$ are everywhere eigenvectors of the shape operator A of f. Set $v_j = \|\partial_j\|$, and define $V_j \in C^\infty(M)$, $1 \leq j \leq n$, by $A\partial_j = v_j^{-1} V_j \partial_j$. Thus, the first and second fundamental forms of f are

$$I = \sum_{i=1}^n v_i^2 du_i^2 \quad \text{and} \quad II = \sum_{i=1}^n V_i v_i du_i^2. \tag{1}$$

Set $v = (v_1, \ldots, v_n)$ and $V = (V_1, \ldots, V_n)$. We call (v, V) the pair associated to f. The next result is well known.

Proposition 5 *The triple* (v, h, V), *where* $h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i}$, *satisfies the system of PDE's*

$$\begin{cases} (i)\dfrac{\partial v_i}{\partial u_j} = h_{ji}v_j, \quad (ii)\dfrac{\partial h_{ik}}{\partial u_j} = h_{ij}h_{jk}, \\ (iii)\dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + h_{ki}h_{kj} + \epsilon_s V_i V_j + c v_i v_j = 0, \\ (iv)\dfrac{\partial V_i}{\partial u_j} = h_{ji}V_j, \quad 1 \le i \ne j \ne k \ne i \le n. \end{cases} \tag{2}$$

Conversely, if (v, h, V) *is a solution of* (2) *on a simply connected open subset* $U \subset \mathbb{R}^n$, *with* $v_i \neq 0$ *everywhere for all* $1 \le i \le n$, *then there exists a holonomic hypersurface* $f: U \to \mathbb{Q}_s^{n+1}(c)$ *whose first and second fundamental forms are given by* (1).

The following characterization of hypersurfaces $f: M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures that are solutions of Problem $*$ is one of the main results of the paper.

Theorem 6 *Let* $f: M^3 \to \mathbb{Q}_s^4(c)$ *be a simply connected holonomic hypersurface whose associated pair* (v, V) *satisfies*

$$\sum_{i=1}^{3} \delta_i v_i^2 = \hat{\epsilon}, \quad \sum_{i=1}^{3} \delta_i v_i V_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = C := \tilde{\epsilon}(c - \tilde{c}), \tag{3}$$

where $\hat{\epsilon}, \tilde{\epsilon} \in \{-1, 1\}$, $\tilde{c} \neq c$, $\hat{\epsilon}\tilde{\epsilon} = \epsilon_s$, $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ *either if* $\hat{\epsilon} = 1$ *or if* $\hat{\epsilon} = -1$ *and* $C > 0$, *and* $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$ *if* $\hat{\epsilon} = -1$ *and* $C < 0$. *Then* M^3 *admits an isometric immersion into* $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, *with* $\epsilon_{\tilde{s}} = \tilde{\epsilon}$, *which is unique up to congruence.*

Conversely, if $f: M^3 \to \mathbb{Q}_s^4(c)$ *is a hypersurface with three distinct principal curvatures for which there exists an isometric immersion* $\tilde{f}: M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ *with* $\tilde{c} \neq c$, *then* f *is locally a holonomic hypersurface whose associated pair* (v, V) *satisfies* (3), *with* $\tilde{\epsilon} = \epsilon_{\tilde{s}}$.

As we shall make precise in the sequel, the class of hypersurfaces that are solutions of Problem $*$ is closely related to that of conformally flat hypersurfaces of $\mathbb{Q}_s^{n+1}(c)$, that is, isometric immersions $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ of conformally flat manifolds. Recall that a Riemannian manifold M^n is *conformally flat* if each point of M^n has an open neighborhood that is conformally diffeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. First, for $n \ge 4$ we have the following extension of a result due to E. Cartan when $s = 0$.

Proposition 7 *Let* $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ *be a hypersurface of dimension* $n \ge 4$. *Then* M^n *is conformally flat if and only if* f *has a principal curvature of multiplicity at least* $n - 1$.

It was already known by E. Cartan that the "only if" assertion in the preceding result is no longer true for $n = 3$ and $s = 0$. The study of conformally flat hypersurfaces by Cartan was taken up by Hertrich-Jeromin [6], who showed that a conformally flat hypersurface $f: M^3 \to \mathbb{Q}^4(c)$ with three distinct principal curvatures admits locally principal coordinates (u_1, u_2, u_3) such that the induced metric $ds^2 = \sum_{i=1}^{3} v_i^2 du_i^2$ satisfies, say, $v_2^2 = v_1^2 + v_3^2$. The next result states that conformally flat hypersurfaces $f: M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures are characterized by the existence of such principal coordinates under some additional conditions.

Theorem 8 *Let* $f: M^3 \to \mathbb{Q}_s^4(c)$ *be a holonomic hypersurface whose associated pair* (v, V) *satisfies*

$$\sum_{i=1}^{3} \delta_i v_i^2 = 0, \quad \sum_{i=1}^{3} \delta_i v_i V_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = 1, \tag{4}$$

where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. *Then* M^3 *is conformally flat, and* f *has three distinct principal curvatures.*

Conversely, any conformally flat hypersurface $f : M^3 \to \mathbb{Q}_s^4(c)$ *with three distinct principal curvatures is locally a holonomic hypersurface whose associated pair* (v, V) *satisfies* (4).

It follows from Theorems 6 and 8 that, in order to produce hypersurfaces of $\mathbb{Q}_s^4(c)$ that are either conformally flat or admit an isometric immersion into $\mathbb{Q}_s^4(\tilde{c})$ with $\tilde{c} \neq c$, one must start with solutions (v, h, V) on an open simply connected subset $U \subset \mathbb{R}^3$ of the same system of PDE's, namely, the one obtained by adding to system (2) (for $n = 3$) the equations

$$\delta_i \frac{\partial v_i}{\partial u_i} + \delta_j h_{ij} v_j + \delta_k h_{ik} v_k = 0 \tag{5}$$

and

$$\delta_i \frac{\partial V_i}{\partial u_i} + \delta_j h_{ij} V_j + \delta_k h_{ik} V_k = 0, \quad 1 \leq i \neq j \neq k \neq i \leq 3, \tag{6}$$

with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. Such system has the first integrals

$$\sum_{i=1}^{3} \delta_i v_i^2 = K_1, \quad \sum_{i=1}^{3} \delta_i v_i V_i = K_2 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = K_3.$$

If initial conditions at some point are chosen so that $K_1 = 1$ (respectively, $K_1 = 0$), $K_2 = 0$ and $K_3 = \epsilon(c - \tilde{c})$ (respectively, $K_3 = 1$), then the corresponding solutions give rise to hypersurfaces of $\mathbb{Q}_s^4(c)$, $\epsilon_s = \epsilon$, with three distinct principal curvatures that can be isometrically immersed into $\mathbb{Q}_s^4(\tilde{c})$ (respectively, are conformally flat).

It was already shown in [5] for $s = 0 = \tilde{s}$ that, unlike the case of dimension $n \geq 4$, among hypersurfaces $f : M^n \to \mathbb{Q}_s^{n+1}(c)$ of dimension $n = 3$ with three distinct principal curvatures the classes of solutions of Problem $*$ and conformally flat hypersurfaces are distinct. Moreover, it was observed that their intersection contains the generalized cones over surfaces with constant curvature in an umbilical hypersurface $\mathbb{Q}_s^3(\bar{c})$ of $\mathbb{Q}_s^4(c)$, $\bar{c} \geq c$. The following result states that such intersection contains no other elements.

Proposition 9 *Let* $f : M^3 \to \mathbb{Q}_s^4(c)$ *be a conformally flat hypersurface with three distinct principal curvatures. If* M^3 *admits an isometric immersion into* $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, $\tilde{c} \neq c$, *then* f *is a generalized cone over a surface with constant curvature in an umbilical hypersurface* $\mathbb{Q}_s^3(\bar{c})$ *of* $\mathbb{Q}_s^4(c)$, $\bar{c} \geq c$.

Our last result shows that hypersurfaces $f : M^3 \to \mathbb{Q}_s^4(c)$ that can be isometrically immersed into $\mathbb{R}_{\tilde{s}}^4$ arise in families of parallel hypersurfaces.

Proposition 10 *Let* $f : M^3 \to \mathbb{Q}_s^4(c)$ *be a holonomic hypersurface whose associated pair* (v, V) *satisfies* (3) *with* $\tilde{c} = 0$. *Then any parallel hypersurface* $f_t : M^3 \to \mathbb{Q}_s^4(c)$ *to* f *has also the same property.*

In a forthcoming paper [2] we develop a Ribaucour transformation for the class of hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures that can be isometrically immersed into $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $c \neq \tilde{c}$, as well as for the class of conformally flat hypersurfaces with three distinct principal curvatures. It gives a process to generate a family of new elements of such classes, starting from a given one and a solution of a linear system of PDE. In particular, explicit new examples of hypersurfaces in both classes are constructed.

1 The proofs

1.1 Proof of Proposition 1

Let $i\colon \mathbb{Q}_s^{n+1}(c) \to \mathbb{Q}_{s+\epsilon_0}^{n+2}(\tilde{c})$ be an umbilical inclusion, where $\epsilon_0 = 0$ or 1, corresponding to $c > \tilde{c}$ or $c < \tilde{c}$, respectively, and set $\hat{f} = i \circ f$. Then, the second fundamental forms α and $\hat{\alpha}$ of f and $\hat{f}$, respectively, are related by

$$\hat{\alpha} = i_*\alpha + \sqrt{|c - \tilde{c}|}\langle\ ,\ \rangle\xi, \tag{7}$$

where ξ is one of the unit vector fields that are normal to i.

For a fixed point $x \in M^n$, define $W^3(x) := N_{\hat{f}}M(x) \oplus N_{\tilde{f}}M(x)$, and endow $W^3(x)$ with the inner product

$$\langle\langle(\xi + \tilde{\xi}, \eta + \tilde{\eta})\rangle\rangle_{W^3(x)} := \langle\xi, \eta\rangle_{N_{\hat{f}}M(x)} - \langle\tilde{\xi}, \tilde{\eta}\rangle_{N_{\tilde{f}}M(x)},$$

which has index $(s + \epsilon_0) + (1 - \tilde{s})$.

Now define a bilinear form $\beta_x\colon T_xM \times T_xM \to W^3(x)$ by

$$\beta_x = \hat{\alpha}(x) \oplus \tilde{\alpha}(x),$$

where $\hat{\alpha}(x)$ and $\tilde{\alpha}(x)$ are the second fundamental forms of $\hat{f}$ and $\tilde{f}$, respectively, at x. Notice that $\mathcal{N}(\beta_x) \subset \mathcal{N}(\hat{\alpha}(x)) = \{0\}$ by (7). On the other hand, it follows from the Gauss equations of $\hat{f}$ and $\tilde{f}$ that β_x is flat with respect to $\langle\langle\ ,\ \rangle\rangle$, that is,

$$\langle\langle\beta_x(X, Y), \beta_x(Z, W)\rangle\rangle = \langle\langle\beta_x(X, W), \beta_x(Z, Y)\rangle\rangle$$

for all $X, Y, Z, W \in T_xM$. Thus, if $\langle\langle\ ,\ \rangle\rangle$ is positive definite, which is the case when $s = 0$, $\tilde{s} = 1$ and $\epsilon_0 = 0$, that is, $c > \tilde{c}$, we obtain a contradiction with Corollary 1 of [7], according to which one has the inequality

$$\dim \mathcal{N}(\beta_x) \geq n - \dim W(x) = n - 3 > 0. \tag{8}$$

The same contradiction is reached by applying the preceding inequality to $-\langle\langle\ ,\ \rangle\rangle$ when $s = 1$, $\tilde{s} = 0$ and $c < \tilde{c}$, in which case $\langle\langle\ ,\ \rangle\rangle$ is negative definite. Therefore, such cases cannot occur, which proves the first assertion.

In all other cases, the index of $\langle\langle\ ,\ \rangle\rangle$ is either 1 or 2. Thus, by applying Corollary 2 in [7] to $\langle\langle\ ,\ \rangle\rangle$ in the first case and to $-\langle\langle\ ,\ \rangle\rangle$ in the latter, we obtain that $\mathcal{S}(\beta_x)$ must be degenerate, for otherwise the inequality (8) would still hold, and then we would reach a contradiction as before.

Since $\mathcal{S}(\beta_x)$ is degenerate, there exist $\zeta \in N_{\hat{f}}M(x)$ and $\tilde{N} \in N_{\tilde{f}}M(x)$ such that $(0, 0) \neq (\zeta, \tilde{N}) \in \mathcal{S}(\beta_x) \cap \mathcal{S}(\beta_x)^{\perp}$. In particular, from $0 = \langle\langle\zeta + \tilde{N}, \zeta + \tilde{N}\rangle\rangle$ it follows that $\langle\tilde{N}, \tilde{N}\rangle = \langle\zeta, \zeta\rangle$. Thus, either $\tilde{N} = 0$ and $\zeta \in \mathcal{S}(\hat{\alpha}(x)) \cap \mathcal{S}(\hat{\alpha}(x))^{\perp}$, or we can assume that $\langle\tilde{N}, \tilde{N}\rangle = \epsilon_{\tilde{s}} = \langle\zeta, \zeta\rangle$.

The former case occurs precisely when f is umbilical at x with a principal curvature λ with respect to one of the unit normal vectors N to f, satisfying

$$\epsilon_s\lambda^2 + c - \tilde{c} = 0,$$

in which case $N_{\hat{f}}M(x)$ is a Lorentzian two-plane and $\zeta = \lambda i_*N + \sqrt{|c - \tilde{c}|}\xi$ is a light-like vector that spans $\mathcal{S}(\hat{\alpha}(x))$. In this case, all sectional curvatures of M^n at x are equal to $\tilde{c}$ by the Gauss equation of f, and hence $\tilde{f}$ has 0 as a principal curvature at x with multiplicity at least $n - 1$ by the Gauss equation of $\tilde{f}$.

Now assume that $\langle \tilde{N}, \tilde{N}\rangle = \epsilon_{\tilde{s}} = \langle \zeta, \zeta\rangle$. Then, from

$$0 = \langle\langle \beta, \zeta + \tilde{N}\rangle\rangle = \langle \hat{\alpha}, \zeta\rangle - \langle \tilde{\alpha}, \tilde{N}\rangle,$$

we obtain that $A^{\hat{f}}_{\zeta} = A^{\tilde{f}}_{\tilde{N}}$. Let $\zeta^{\perp} \in N_{\hat{f}}M(x)$ be such that $\{\zeta, \zeta^{\perp}\}$ is an orthonormal basis of $N_{\hat{f}}M(x)$. The Gauss equations for $\hat{f}$ and $\tilde{f}$ imply that

$$\langle A^{\hat{f}}_{\zeta^{\perp}} X, Y\rangle \langle A^{\hat{f}}_{\zeta^{\perp}} Z, W\rangle = \langle A^{\hat{f}}_{\zeta^{\perp}} X, W\rangle \langle A^{\hat{f}}_{\zeta^{\perp}} Z, Y\rangle$$

for all $X, Y, Z, W \in T_xM$, which is equivalent to $\dim \mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}}) \geq n-1$. Since $A^{\hat{f}}_{\xi} = \delta\sqrt{|c-\tilde{c}|}I$ by (7), with $\delta = (c-\tilde{c})/|c-\tilde{c}|$, it follows that the restriction to $\mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}})$ of all shape operators $A^{\hat{f}}_{\eta}$, $\eta \in N_{\hat{f}}M(x)$, is a multiple of the identity tensor. In particular, this is the case for $A^{\hat{f}}_{i_*N} = A^{f}_{N}$, where N is one of the unit normal vector fields to f, hence f has a principal curvature λ at x with multiplicity at least $n-1$.

Moreover, if $\lambda = 0$ then $\zeta^{\perp}$ must coincide with i_*N, and hence ζ with ξ, up to signs. Therefore $A^{\tilde{f}}_{\tilde{N}} = A^{\hat{f}}_{\xi}$, up to sign, hence $\tilde{f}$ is umbilical at x. If f is umbilical at x and $c+\epsilon_s\lambda^2 \neq \tilde{c}$, then $A_{\zeta^{\perp}} = 0$ and $A^{\tilde{f}}_{\tilde{N}} = A^{\hat{f}}_{\zeta}$ is a (nonzero) constant multiple of the identity tensor. Finally, if $\lambda \neq 0$ has multiplicity $n-1$, then we must have $\zeta^{\perp} \neq i_*N$ and $\dim \mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}}) = n-1$, hence $\mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}})$ is an eigenspace of $A^{\hat{f}}_{\zeta} = A^{\tilde{f}}_{\tilde{N}}$. □

1.2 Proof of Proposition 2

Suppose first that f is umbilical, with a (constant) principal curvature λ. If $\rho = 0$, then M^n has constant curvature $\tilde{c}$, hence it admits isometric immersions into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$ having 0 as a principal curvature with multiplicity at least $n-1$. If $\rho > 0$, there exists $\tilde{\lambda} \neq 0$ such that $c - \tilde{c} + \epsilon_s\lambda^2 = \epsilon_{\tilde{s}}\tilde{\lambda}^2$. Hence $c + \epsilon_s\lambda^2 = \tilde{c} + \epsilon_{\tilde{s}}\tilde{\lambda}^2$, thus $\tilde{A} = \tilde{\lambda}I$ satisfies the Gauss and Codazzi equations for an (umbilical) isometric immersion into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$.

Assume now that λ has constant multiplicity $n-1$. If $\lambda = 0$, then M^n has constant curvature c and by the assumption there exists $\tilde{\lambda} \neq 0$ such that $c = \tilde{c} + \epsilon_{\tilde{s}}\tilde{\lambda}^2$. Thus, $\tilde{A} = \tilde{\lambda}I$ satisfies the Gauss and Codazzi equations for an (umbilical) isometric immersion into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$.

From now on assume that $\lambda \neq 0$. Let μ be the simple principal curvature and let E_λ and E_μ denote the corresponding eigenbundles. Then, one can check that the Codazzi equations for f are equivalent to the following facts:

(i) λ is constant along E_λ;
(ii) E_λ is an umbilical distribution whose mean curvature vector field is $\eta = (\lambda-\mu)^{-1}\nabla\lambda$;
(iii) The mean curvature vector field (geodesic curvature vector field) of E_μ is $\zeta = (\mu - \lambda)^{-1}(\nabla\mu)_{E_\lambda}$.

By the assumption, there exist $\tilde{\lambda}, \tilde{\mu} \in C^{\infty}(M)$ such that

$$c - \tilde{c} + \epsilon_s\lambda^2 = \epsilon_{\tilde{s}}\tilde{\lambda}^2 \quad \text{and} \quad c - \tilde{c} + \epsilon_s\lambda\mu = \epsilon_{\tilde{s}}\tilde{\lambda}\tilde{\mu}.$$

Moreover, since $\lambda \neq \mu$ we must have $\tilde{\lambda} \neq 0$ everywhere, and hence $\tilde{\lambda}$ and $\tilde{\mu}$ are unique if $\tilde{\lambda}$ is chosen to be positive. From both equations we obtain

$$\epsilon_s\lambda^2 - \epsilon_{\tilde{s}}\tilde{\lambda}^2 = \epsilon_s\lambda\mu - \epsilon_{\tilde{s}}\tilde{\lambda}\tilde{\mu}, \quad \epsilon_s\lambda\nabla\lambda = \epsilon_{\tilde{s}}\tilde{\lambda}\nabla\tilde{\lambda}$$

and

$$\epsilon_s\left((\nabla\lambda)\mu+\lambda\nabla\mu\right)=\epsilon_{\tilde{s}}\left(\left(\nabla\tilde{\lambda}\right)\tilde{\mu}+\tilde{\lambda}\nabla\tilde{\mu}\right).$$

It follows that

$$\frac{\nabla\tilde{\lambda}}{\tilde{\lambda}-\tilde{\mu}}=\frac{\nabla\lambda}{\lambda-\mu} \tag{9}$$

and similarly,

$$\frac{(\nabla\tilde{\mu})_{E_\lambda}}{\tilde{\mu}-\tilde{\lambda}}=\frac{(\nabla\mu)_{E_\lambda}}{\mu-\lambda}. \tag{10}$$

Let $\tilde{A}$ be the endomorphism of TM with eigenvalues $\tilde{\lambda}$, $\tilde{\mu}$ and corresponding eigenbundles E_λ and E_μ, respectively. Since

$$c+\epsilon_s\lambda^2=\tilde{c}+\epsilon_{\tilde{s}}\tilde{\lambda}^2 \quad\text{and}\quad c+\epsilon_s\lambda\mu=\tilde{c}+\epsilon_{\tilde{s}}\tilde{\lambda}\tilde{\mu},$$

the Gauss equations for an isometric immersion $\tilde{f}\colon M^n\to\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ are satisfied by $\tilde{A}$. It follows from (9) and (10) that $\tilde{A}$ also satisfies the Codazzi equations. □

1.3 Proof of Proposition 3

Since $c>\tilde{c}$, there exist umbilical inclusions $i\colon\mathbb{Q}_s^{n+1}(c)\to\mathbb{Q}_s^{n+2}(\tilde{c})$ and $i\colon\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})\to\mathbb{Q}_s^{n+2}(c)$ for $(s,\tilde{s})=(1,0)$. If $s=\tilde{s}$ (respectively, $(s,\tilde{s})=(1,0)$), set $\hat{f}=i\circ f$ (respectively, $\hat{f}=i\circ\tilde{f}$). Then, one can use the existence of normal vector fields $\zeta\in\Gamma(N_{\hat{f}}M)$ and $\tilde{N}\in\Gamma(N_{\tilde{f}}M)$ satisfying $\langle\zeta,\zeta\rangle=\tilde{\epsilon}=\langle\tilde{N},\tilde{N}\rangle$ and $A_\zeta^{\hat{f}}=A_{\tilde{N}}^{\tilde{f}}$ and argue as in the proof of Theorem 3 in [5]. One obtains that there exists an open dense subset $U\subset M^n$, each point of which has an open neighborhood $V\subset M^n$ such that $\hat{f}|_V$ (respectively, $f|_V$) is a composition $\hat{f}|_V=H\circ\tilde{f}|_V$ (respectively, $f|_V=H\circ\hat{f}|_V$) with an isometric embedding $H\colon W\subset\mathbb{Q}_s^{n+1}(\tilde{c})\to\mathbb{Q}_s^{n+2}(\tilde{c})$ (respectively, $H\colon W\subset\mathbb{Q}_s^{n+1}(c)\to\mathbb{Q}_s^{n+2}(c)$), with $\tilde{f}(V)\subset W$ (respectively, $\hat{f}(V)\subset W$). Set $\bar{M}^n=H(W)\cap i(\mathbb{Q}_s^{n+1}(c))$ (respectively, $\bar{M}^n=H(W)\cap i(\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c}))$). Then $i\circ f|_V=H\circ\tilde{f}|_V\colon V\to\bar{M}^n$ (respectively, $H\circ f|_V=i\circ\tilde{f}|_V\colon V\to\bar{M}^n$) is an isometry. Let $\Psi\colon\bar{M}^n\to V$ be the inverse of this isometry. Then $f\circ\Psi=i^{-1}|_{\bar{M}^n}$ and $\tilde{f}\circ\Psi=H^{-1}|_{\bar{M}^n}$ (respectively, $f\circ\Psi=H^{-1}|_{\bar{M}^n}$ and $\tilde{f}\circ\Psi=i^{-1}|_{\bar{M}^n}$), where i^{-1} and H^{-1} denote the inverses of the maps i and H, respectively, regarded as maps onto their images. □

1.4 Proof of Theorem 4

Before going into the proof of Theorem 4, we establish a basic fact that will also be used in the proof of Theorem 6 in the next section.

Lemma 11 *Let $f\colon M^3\to\mathbb{Q}_s^4(c)$ and $\tilde{f}\colon M^3\to\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ be isometric immersions with $c\neq\tilde{c}$. Then, at each point $x\in M^3$ there exists an orthonormal basis $\{e_1,e_2,e_3\}$ of T_xM^3 that simultaneously diagonalizes the second fundamental forms of f and $\tilde{f}$.*

Proof Define $i\colon\mathbb{Q}_s^4(c)\to\mathbb{Q}_{s+\epsilon_0}^5(\tilde{c})$ and $\hat{f}$, as well as $W^3(x)$, $\langle\!\langle\ ,\ \rangle\!\rangle_{W^3(x)}$ and β_x for each $x\in M^n$, as in the proof of Proposition 1. If $\mathcal{S}(\beta_x)$ is degenerate for all $x\in M^3$, we conclude as in the case $n \geq 4$ that the assertions in Theorem 1 hold, hence the statement is clearly true in this case.

Suppose now that $\mathcal{S}(\beta_x)$ is nondegenerate at $x \in M^3$. Then the inequality

$$\dim \mathcal{S}(\beta_x) \geq \dim T_x M - \dim \mathcal{N}(\beta_x)$$

holds by Corollary 2 in [7]. Since $\mathcal{N}(\beta_x) = \{0\}$, the right-hand side is equal to $\dim T_x M = 3 = \dim W^3(x)$, hence we must have equality in the above inequality. By Theorem $2-b$ in [7], there exists an orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ of $W^3(x)$ and a basis $\{\theta^1, \theta^2, \theta^3\}$ of $T_x^* M$ such that

$$\beta = \sum_{j=1}^{3} \theta^j \otimes \theta^j \xi_j.$$

In particular, if $i \neq j$ then $\beta(e_i, e_j) = 0$ for the dual basis $\{e_1, e_2, e_3\}$ of $\{\theta^1, \theta^2, \theta^3\}$. It follows that $\{e_1, e_2, e_3\}$ diagonalyzes both $\hat{\alpha}$ and $\tilde{\alpha}$, and therefore both α and $\tilde{\alpha}$, in view of (7). It also follows from (7) that

$$0 = \langle \hat{\alpha}(e_i, e_j), \xi \rangle = \sqrt{|c - \tilde{c}|} \langle e_i, e_j \rangle, \quad i \neq j,$$

hence the basis $\{e_1, e_2, e_3\}$ is orthogonal. □

Lemma 12 *Under the assumptions of Lemma 11, let $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 be the principal curvatures of f and $\tilde{f}$ correspondent to e_1, e_2 and e_3, respectively.*

(a) *Assume that f has a principal curvature of multiplicity two, say, that $\lambda_1 = \lambda_2 := \lambda$. If either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, then*

$$c - \tilde{c} + \epsilon_s \lambda \lambda_3 = 0, \quad \mu_3 = 0 \quad \text{and} \quad c - \tilde{c} + \epsilon_s \lambda^2 = \epsilon_{\tilde{s}} \mu_1 \mu_2.$$

Otherwise, either the same conclusion holds or

$$\mu_1 = \mu_2 := \mu, \quad c - \tilde{c} + \epsilon_s \lambda^2 = \epsilon_{\tilde{s}} \mu^2 \quad \text{and} \quad c - \tilde{c} + \epsilon_s \lambda \lambda_3 = \epsilon_{\tilde{s}} \mu \mu_3.$$

(b) *Assume, say, that $\lambda_3 = 0$. Then $\mu_1 = \mu_2 := \mu$,*

$$c - \tilde{c} + \epsilon_s \lambda_1 \lambda_2 = \epsilon_{\tilde{s}} \mu^2 \tag{11}$$

and

$$c - \tilde{c} = \epsilon_{\tilde{s}} \mu \mu_3. \tag{12}$$

Proof By the Gauss equations for f and $\tilde{f}$, we have

$$c + \epsilon_s \lambda_i \lambda_j = \tilde{c} + \epsilon_{\tilde{s}} \mu_i \mu_j, \quad 1 \leq i \neq j \leq 3. \tag{13}$$

(a) If $\lambda_1 = \lambda_2 := \lambda$, then the preceding equations are

$$c + \epsilon_s \lambda^2 = \tilde{c} + \epsilon_{\tilde{s}} \mu_1 \mu_2, \tag{14}$$

$$c + \epsilon_s \lambda \lambda_3 = \tilde{c} + \epsilon_{\tilde{s}} \mu_1 \mu_3 \tag{15}$$

and

$$c + \epsilon_s \lambda \lambda_3 = \tilde{c} + \epsilon_{\tilde{s}} \mu_2 \mu_3. \tag{16}$$

The two last equations yield

$$\mu_3(\mu_1 - \mu_2) = 0,$$

hence either $\mu_3 = 0$ or $\mu_1 = \mu_2$. In view of (14), the second possibility cannot occur if either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$. Thus, in these cases we must have $\mu_3 = 0$, and then $c - \tilde{c} + \epsilon_s \lambda^2 = \epsilon_{\tilde{s}}\mu_1\mu_2$ and $c - \tilde{c} + \epsilon_s \lambda\lambda_3 = 0$ by (15) and (16).

Otherwise, either the same conclusion holds or $\mu_1 = \mu_2 := \mu$, and then $c - \tilde{c} + \epsilon_s\lambda^2 = \epsilon_{\tilde{s}}\mu^2$ and $c - \tilde{c} + \epsilon_s\lambda\lambda_3 = \epsilon_{\tilde{s}}\mu\mu_3$ by (15) and (16).

(b) If $\lambda_3 = 0$, then Eq. (13) become

$$c - \tilde{c} + \epsilon_s\lambda_1\lambda_2 = \epsilon_{\tilde{s}}\mu_1\mu_2, \tag{17}$$

$$c - \tilde{c} = \epsilon_{\tilde{s}}\mu_1\mu_3 \tag{18}$$

and

$$c - \tilde{c} = \epsilon_{\tilde{s}}\mu_2\mu_3 \tag{19}$$

Since $\mu_3 \neq 0$ by (18) or (19), these equations imply that $\mu_1 = \mu_2 := \mu$, and we obtain (12). Equation (11) then follows from (17). □

Proof of Theorem 4 Assume that f has a principal curvature of multiplicity two, say, $\lambda_1 = \lambda_2 := \lambda$. Suppose first that either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$. Then, it follows from Lemma 12 that

$$c - \tilde{c} + \epsilon_s\lambda\lambda_3 = 0, \quad \mu_3 = 0 \quad \text{and} \quad c - \tilde{c} + \epsilon_s\lambda^2 = \epsilon_{\tilde{s}}\mu_1\mu_2. \tag{20}$$

In particular, we must have $\lambda \neq 0$ by the first of the preceding equations, whereas the last one implies that $\mu_1\mu_2 \neq 0$. Then, it is well known that E_λ is a spherical distribution, that is, it is umbilical and its mean curvature normal $\eta = \nu e_3$ satisfies $e_1(\nu) = 0 = e_2(\nu)$. In particular, a leaf σ of E_λ has constant sectional curvature $\nu^2 + \epsilon_s\lambda^2 + c = \nu^2 + \epsilon_{\tilde{s}}\mu_1\mu_2 + \tilde{c}$. Denoting by ∇ and $\tilde{\nabla}$ the connections on M^3 and $\tilde{f}^*T\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$, respectively, we have

$$\tilde{\nabla}_{e_i}\tilde{f}_*e_3 = \tilde{f}_*\nabla_{e_i}e_3 = -\nu\tilde{f}_*e_i, \quad 1 \leq i \leq 2,$$

hence $\tilde{f}(\sigma)$ is contained in an umbilical hypersurface $\mathbb{Q}^3_{\tilde{s}}(\bar{c})$ of $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ with constant curvature $\bar{c} = \tilde{c} + \nu^2$ and $\tilde{f}_*e_3$ as a unit normal vector field.

Moreover, $E_\lambda^\perp = E_{\mu_3}$ is the relative nullity distribution of $\tilde{f}$. Thus, it is totally geodesic, and in fact its integral curves are mapped by $\tilde{f}$ into geodesics of $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$. It follows that $\tilde{f}(M^3)$ is contained in a generalized cone over $\tilde{f}(\sigma)$.

On the other hand, it is not hard to extend the proof of Theorem 4.2 in [4] to the case of Lorentzian ambient space forms, and conclude that f is a rotation hypersurface in $\mathbb{Q}^4_s(c)$. This means that there exist subspaces $P^2 \subset P^3 = P^3_{s+\epsilon_0}$ in $\mathbb{R}^5_{s+\epsilon_0} \supset \mathbb{Q}^4_s(c)$ with $P^3 \cap \mathbb{Q}^4_s(c) \neq \emptyset$, where $\epsilon_0 = 0$ or $\epsilon_0 = 1$, corresponding to $c > 0$ or $c < 0$, respectively, and a regular curve γ in $\mathbb{Q}^2_s(c) = P^3 \cap \mathbb{Q}^4_s(c)$ that does not meet P^2, such that $f(M^2)$ is the union of the orbits of points of γ under the action of the subgroup of orthogonal transformations of $\mathbb{R}^5_{s+\epsilon_0}$ that fix pointwise P^2. If P^2 is nondegenerate, then f can be parameterized by

$$f(s,u) = (\gamma_1(s)\phi_1(u), \gamma_1(s)\phi_2(u), \gamma_1(s)\phi_3(u), \gamma_4(s), \gamma_5(s)),$$

with respect to an orthonormal basis $\{e_1, \ldots, e_5\}$ of $\mathbb{R}^5_{s+\epsilon_0}$ satisfying the conditions in either (i) or (ii) below, according to whether the induced metric on P^2 has index $s+\epsilon_0$ or $s+\epsilon_0-1$, respectively:

(i) $\langle e_i, e_i\rangle = 1$ for $1 \leq i \leq 3$, $\langle e_{3+j}, e_{3+j}\rangle = \epsilon_j$ for $1 \leq j \leq 2$, and (ϵ_1, ϵ_2) equal to either $(1,1)$, $(1,-1)$ or $(-1,-1)$, corresponding to $s + \epsilon_0 = 0$, 1 or 2, respectively.

(ii) $\langle e_1, e_1\rangle = -1$, $\langle e_i, e_i\rangle = 1$ for $2 \le i \le 4$ and $\langle e_5, e_5\rangle = \bar\epsilon$, where $\bar\epsilon = 1$ or $\bar\epsilon = -1$, corresponding to $s + \epsilon_0 = 1$ or 2, respectively.

In both cases, we have $P^2 = \operatorname{span}\{e_4, e_5\}$, $P^3 = \operatorname{span}\{e_1, e_4, e_5\}$, $u = (u_1, u_2)$, $\gamma(s) = (\gamma_1(s), \gamma_4(s), \gamma_5(s))$ a unit–speed curve in $\mathbb{Q}^2_s(c) \subset P^3$ and $\phi(u) = (\phi_1(u), \phi_2(u), \phi_3(u))$ an orthogonal parameterization of the unit sphere $\mathbb{S}^2 \subset (P^2)^\perp$ in case (i) and of the hyperbolic plane $\mathbb{H}^2 \subset (P^2)^\perp$ in case (ii). Accordingly, the hypersurface is said to be of spherical or hyperbolic type.

If P^2 is degenerate, then f is a rotation hypersurface of parabolic type parameterized by

$$f(s, u) = \left(\gamma_1(s), \gamma_1(s)u_1, \gamma_1(s)u_2, \gamma_4(s) - \frac{1}{2}\gamma_1(s)(u_1^2 + u_2^2), \gamma_5(s)\right),$$

with respect to a pseudo-orthonormal basis $\{e_1, \ldots, e_5\}$ of $\mathbb{R}^5_{s+\epsilon_0}$ such that $\langle e_1, e_1\rangle = 0 = \langle e_4, e_4\rangle$, $\langle e_1, e_4\rangle = 1$, $\langle e_2, e_2\rangle = 1 = \langle e_3, e_3\rangle$ and $\langle e_5, e_5\rangle = -2(s + \epsilon_0) + 3$, where $\gamma(s) = (\gamma_1(s), \gamma_4(s), \gamma_5(s))$ is a unit–speed curve in $\mathbb{Q}^2_s(c) \subset P^3 = \operatorname{span}\{e_1, e_4, e_5\}$.

In each case, one can compute the principal curvatures of f as in [4] and check that the first equation in (20) is satisfied if and only if $\gamma_1'' + \tilde c\gamma_1 = 0$, that is, γ is a $\tilde c$-helix in $\mathbb{Q}^2_s(c) \subset \mathbb{R}^3_{s+\epsilon_0}$.

Under the remaining possibilities for c, $\tilde c$, s and $\tilde s$, either the same conclusions hold or the bilinear form β_x in the proof of Proposition 1 is everywhere degenerate, in which case there exist normal vector fields $\zeta \in \Gamma(N_{\hat f}M)$ and $\tilde N \in \Gamma(N_{\tilde f}M)$ satisfying $\langle \zeta, \zeta\rangle = \epsilon_{\tilde s} = \langle \tilde N, \tilde N\rangle$ and $A^{\hat f}_\zeta = A^{\tilde f}_{\tilde N}$, and we obtain as before that f and $\tilde f$ are locally given on an open dense subset as described in Proposition 3.

Finally, if one of the principal curvatures of f is zero, then the preceding argument applies with the roles of f and $\tilde f$ interchanged. □

1.5 Proof of Theorem 6

Let (v, V) be the pair associated to f. Define

$$\tilde V_j = (-1)^{j+1}\delta_j(v_i V_k - v_k V_i), \quad 1 \le i \ne j \ne k \le 3, \quad i < k. \tag{21}$$

Then $\tilde V = (\tilde V_1, \tilde V_2, \tilde V_3)$ is the unique vector in $\mathbb{R}^3$, up to sign, such that $(v, |C|^{-1/2}V, |C|^{-1/2}\tilde V)$ is an orthonormal basis of $\mathbb{R}^3$ with respect to the inner product

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = \sum_{i=1}^{3} \delta_i x_i y_i. \tag{22}$$

Therefore, the matrix $D = (v, |C|^{-1/2}V, |C|^{-1/2}\tilde V)$ satisfies $D\delta D^t = \delta$, where $\delta = \operatorname{diag}(\hat\epsilon, C/|C|, -\hat\epsilon C/|C|)$. It follows that

$$\hat\epsilon v_i v_j + C/|C|^2 V_i V_j - \hat\epsilon C/|C|^2 \tilde V_i \tilde V_j = 0, \quad 1 \le i \ne j \le 3.$$

Multiplying by $\epsilon_s C$ and using that $\hat\epsilon\epsilon_s = \tilde\epsilon$ and $\hat\epsilon\epsilon_s C = \hat\epsilon\epsilon_s\tilde\epsilon(c - \tilde c) = c - \tilde c$ we obtain

$$(c - \tilde c)v_i v_j + \epsilon V_i V_j - \tilde\epsilon \tilde V_i \tilde V_j = 0,$$

or equivalently,

$$c v_i v_j + \epsilon V_i V_j = \tilde c v_i v_j + \tilde\epsilon \tilde V_i \tilde V_j.$$

Substituting the preceding equation into (v) yields

$$\frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ji}}{\partial u_j} + h_{ki}h_{kj} + \tilde{\epsilon}\tilde{V}_i\tilde{V}_j + \tilde{c}v_iv_j = 0.$$

On the other hand, differentiating (21) and using equations (i)–(iv) yields

$$\frac{\partial \tilde{V}_j}{\partial u_i} = h_{ij}\tilde{V}_i, \quad 1 \le i \ne j \le 3.$$

It follows from Proposition 5 that there exists a hypersurface $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$, with $\epsilon_{\tilde{s}} = \tilde{\epsilon}$, whose first and second fundamental forms are

$$I = \sum_{i=1}^{3} v_i^2 du_i^2 \quad \text{and} \quad II = \sum_{i=1}^{3} \tilde{V}_i v_i du_i^2,$$

thus M^3 admits an isometric immersion into $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$.

Conversely, let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a hypersurface for which there exists an isometric immersion $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$. By Lemma 11, there exists an orthonormal frame $\{e_1, e_2, e_3\}$ of M^3 of principal directions of both f and $\tilde{f}$. Let $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 be the principal curvatures of f and $\tilde{f}$ correspondent to e_1, e_2 and e_3, respectively. Assume that $\lambda_1 < \lambda_2 < \lambda_3$, and that the unit normal vector field to f has been chosen so that $\lambda_1 < 0$. The Gauss equations for f and $\tilde{f}$ yield

$$c + \epsilon_s\lambda_i\lambda_j = \tilde{c} + \epsilon_{\tilde{s}}\mu_i\mu_j, \quad 1 \le i \ne j \le 3.$$

Thus,

$$\mu_i\mu_j = C + \hat{\epsilon}\lambda_i\lambda_j, \quad C = \epsilon_{\tilde{s}}(c - \tilde{c}), \quad 1 \le i \ne j \le 3. \tag{23}$$

It follows that

$$\mu_j^2 = \frac{(C + \hat{\epsilon}\lambda_j\lambda_i)(C + \hat{\epsilon}\lambda_j\lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}, \quad 1 \le j \ne i \ne k \ne j \le 3. \tag{24}$$

The Codazzi equations for f and $\tilde{f}$ are, respectively,

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\langle \nabla_{e_j} e_i, e_j\rangle, \quad i \ne j, \tag{25}$$

$$(\lambda_j - \lambda_k)\langle \nabla_{e_i} e_j, e_k\rangle = (\lambda_i - \lambda_k)\langle \nabla_{e_j} e_i, e_k\rangle, \quad i \ne j \ne k. \tag{26}$$

and

$$e_i(\mu_j) = (\mu_i - \mu_j)\langle \nabla_{e_j} e_i, e_j\rangle, \; i \ne j, \tag{27}$$

$$(\mu_j - \mu_k)\langle \nabla_{e_i} e_j, e_k\rangle = (\mu_i - \mu_k)\langle \nabla_{e_j} e_i, e_k\rangle, \; i \ne j \ne k. \tag{28}$$

Multiplying (28) by μ_j and using (24) and (26) we obtain

$$\hat{\epsilon}C\frac{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}\langle \nabla_{e_i} e_j, e_k\rangle = 0, \; i \ne j \ne k.$$

Since the principal curvatures λ_1, λ_2 and λ_3 are distinct, it follows that

$$\langle \nabla_{e_i} e_j, e_k\rangle = 0, \quad 1 \le i \ne j \ne k \ne i \le 3. \tag{29}$$

Computing $2\mu_j e_i(\mu_j)$, first by differentiating (24) and then by multiplying (27) by $2\mu_j$, and using (23), (24) and (25) we obtain

$$\begin{aligned}(C+\hat{\epsilon}\lambda_j\lambda_k)(\lambda_k-\lambda_j)e_i(\lambda_i)+(C+\hat{\epsilon}\lambda_i\lambda_k)(\lambda_k-\lambda_i)e_i(\lambda_j)\\ +(C+\hat{\epsilon}\lambda_i\lambda_j)(\lambda_i-\lambda_j)e_i(\lambda_k)=0.\end{aligned} \tag{30}$$

Now let $\{\omega_1,\omega_2,\omega_3\}$ be the dual frame of $\{e_1,e_2,e_3\}$ and define the one-forms γ_j, $1\le j\le 3$, by

$$\gamma_j=\sqrt{\delta_j\frac{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}{C+\hat{\epsilon}\lambda_i\lambda_k}}\omega_j,\quad 1\le j\ne i\ne k\ne j\le 3,$$

where $\delta_j=y_j/|y_j|$ for $y_j=\frac{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}{C+\hat{\epsilon}\lambda_i\lambda_k}$.

By (24), either all the three numbers $C+\hat{\epsilon}\lambda_j\lambda_i$, $C+\hat{\epsilon}\lambda_j\lambda_k$ and $C+\hat{\epsilon}\lambda_i\lambda_k$ are positive or two of them are negative and the remaining one is positive. Hence there are four possible cases:

(I) $C+\hat{\epsilon}\lambda_i\lambda_j>0$, $1\le i\ne j\le 3$.
(II) $C+\hat{\epsilon}\lambda_1\lambda_2<0$, $C+\hat{\epsilon}\lambda_1\lambda_3<0$ and $C+\hat{\epsilon}\lambda_2\lambda_3>0$.
(III) $C+\hat{\epsilon}\lambda_1\lambda_2>0$, $C+\hat{\epsilon}\lambda_1\lambda_3<0$ and $C+\hat{\epsilon}\lambda_2\lambda_3<0$.
(IV) $C+\hat{\epsilon}\lambda_1\lambda_2<0$, $C+\hat{\epsilon}\lambda_1\lambda_3>0$ and $C+\hat{\epsilon}\lambda_2\lambda_3<0$.

Notice that $(\delta_1,\delta_2,\delta_3)$ equals $(1,-1,1)$ in case (I), $(1,1,-1)$ in case (II), $(-1,1,1)$ in case (III) and $(-1,-1,-1)$ in case (IV). It is easily checked that one must have $\hat{\epsilon}=-1$ and $C<0$ in case (IV), whereas in the remaining cases either $\hat{\epsilon}=1$ or $\hat{\epsilon}=-1$ and $C>0$. Therefore, $(\delta_1,\delta_2,\delta_3)=(-1,-1,-1)$ if $\hat{\epsilon}=-1$ and $C<0$, and in the remaining cases we may assume that $(\delta_1,\delta_2,\delta_3)=(1,-1,1)$ after possibly reordering the coordinates.

We claim that (29) and (30) are precisely the conditions for the one-forms γ_j, $1\le j\le 3$, to be closed. To prove this, set $x_j=\sqrt{\delta_j y_j}$, $1\le j\le 3$, so that $\gamma_j=x_j\omega_j$. It follows from (29) that

$$d\gamma_j(e_i,e_k)=e_i\gamma_j(e_k)-e_k\gamma_j(e_i)-\gamma_j([e_i,e_k])=0.$$

On the other hand, using (25) we obtain

$$\begin{aligned}d\gamma_j(e_i,e_j)&=e_i\gamma_j(e_j)-e_j\gamma_j(e_i)-\gamma_j([e_i,e_j])\\&=e_i(x_j)+x_j\langle\nabla_{e_j}e_i,e_j\rangle\\&=e_i(x_j)+x_j\frac{e_i(\lambda_j)}{\lambda_i-\lambda_j},\end{aligned}$$

hence γ_j is closed if and only if

$$e_i(x_j)=\frac{x_j}{\lambda_j-\lambda_i}e_i(\lambda_j),\quad 1\le i\ne j\le 3,$$

or equivalently,

$$\begin{aligned}e_i(y_j)(C+\hat{\epsilon}\lambda_i\lambda_k)&=2\delta_j x_j e_i(x_j)(C+\hat{\epsilon}\lambda_i\lambda_k)\\&=2\delta_j\frac{x_j^2}{\lambda_j-\lambda_i}e_i(\lambda_j)(C+\hat{\epsilon}\lambda_i\lambda_k)\\&=2(\lambda_j-\lambda_k)e_i(\lambda_j).\end{aligned}$$

The preceding equation is in turn equivalent to

$$\begin{aligned}2(\lambda_j-\lambda_k)(C+\hat{\epsilon}\lambda_i\lambda_k)e_i(\lambda_j) &= (e_i(\lambda_j)-e_i(\lambda_i)(\lambda_j-\lambda_k)(C+\hat{\epsilon}\lambda_i\lambda_k)\\ &\quad +(\lambda_j-\lambda_i)(e_i(\lambda_j)-e_i(\lambda_k))(C+\hat{\epsilon}\lambda_i\lambda_k)\\ &\quad -(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)(\hat{\epsilon}(e_i(\lambda_i)\lambda_k+\lambda_i e_i(\lambda_k)),\end{aligned}$$

which is the same as (30).

Therefore, each point $x\in M^3$ has an open neighborhood V where one can find functions $u_j\in C^\infty(V)$, $1\le j\le 3$, such that $du_j=\gamma_j$, and we can choose V so small that $\Phi=(u_1,u_2,u_3)$ is a diffeomorphism of V onto an open subset $U\subset\mathbb{R}^3$, that is, (u_1,u_2,u_3) are local coordinates on V. From $\delta_{ij}=du_j(\partial u_i)=x_j\omega_j(\partial u_i)$ it follows that $\partial u_i=v_ie_i$, with $v_i=x_i^{-1}$. Now notice that

$$\sum_{j=1}^{3}\delta_j v_j^2=\sum_{i,k\neq j=1}^{3}\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=\hat{\epsilon},$$

$$\sum_{j=1}^{3}\delta_j v_j V_j=\sum_{j=1}^{3}\delta_j\lambda_j v_j^2=\sum_{i,k\neq j=1}^{3}\lambda_j\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=0$$

and

$$\sum_{j=1}^{3}\delta_j V_j^2=\sum_{j=1}^{3}\delta_j\lambda_j^2 v_j^2=\sum_{i,k\neq j=1}^{3}\lambda_j^2\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=C.$$

It follows that the pair (v,V) satisfies (3). □

1.6 Proof of Proposition 7

Before starting the proof of Proposition 7, recall that the *Weyl tensor* of a Riemannian manifold M^n is defined by

$$\begin{aligned}\langle C(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle - L(X,W)\langle Y,Z\rangle - L(Y,Z)\langle X,W\rangle\\ &\quad + L(X,Z)\langle Y,W\rangle + L(Y,W)\langle X,Z\rangle\end{aligned}$$

for all $X,Y,Z,W\in\mathfrak{X}(M)$, where L is the *Schouten tensor* of M^n, which is given in terms of the Ricci tensor and the scalar curvature s by

$$L(X,Y)=\frac{1}{n-2}\left(\mathrm{Ric}\,(X,Y)-\frac{1}{2}ns\langle X,Y\rangle\right).$$

It is well known that, if $n\ge 4$, then the vanishing of the Weyl tensor is a necessary and sufficient condition for M^n to be conformally flat.

Proof of Proposition 7 Let $f\colon M^n\to\mathbb{Q}_s^{n+1}(c)$ be a conformally flat hypersurface of dimension $n\ge 4$. For a fixed point $x\in M^n$, choose a unit normal vector $N\in N_x^fM$ and let $A=A_N\colon T_xM\to T_xM$ be the shape operator of f with respect to N. Let W^3 be a vector space endowed with the Lorentzian inner product $\langle\langle\,,\,\rangle\rangle$ given by

$$\langle\langle(a,b,c),(a',b',c')\rangle\rangle=\epsilon(-aa'+bb'+\epsilon cc').$$

Define a bilinear form $\beta\colon T_xM\times T_xM\to W^3$ by

$$\beta(X,Y)=(L(X,Y)+\frac{1}{2}(1-c)\langle X,Y\rangle, L(X,Y)-\frac{1}{2}(1+c)\langle X,Y\rangle, \langle AX,Y\rangle).$$

Note that $\beta(X, X) \neq 0$ for all $X \neq 0$. Moreover,

$$\begin{aligned}&\langle\!\langle \beta(X, Y), \beta(Z, W)\rangle\!\rangle - \langle\!\langle \beta(X, W), \beta(Z, Y)\rangle\!\rangle = -L(X, Y)\langle Z, W\rangle\\ &\quad - L(Z, W)\langle X, Y\rangle + L(X, W)\langle Z, Y\rangle + L(Z, Y)\langle X, W\rangle + c\langle (X \wedge Z)W, Y\rangle\\ &\quad + \epsilon\langle (AX \wedge AZ)W, Y\rangle = \langle C(X, Z)W, Y\rangle = 0.\end{aligned}$$

Thus, β is flat with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. We claim that $S(\beta)$ must be degenerate. Otherwise, we would have

$$0 = \dim\ker\beta \geq n - \dim S(\beta) > 0,$$

a contradiction. Now let $\zeta \in S(\beta) \cap S(\beta)^{\perp}$ and choose a pseudo-orthonormal basis ζ, η, ξ of W^3 with $\langle\!\langle \zeta, \zeta\rangle\!\rangle = 0 = \langle\!\langle \eta, \eta\rangle\!\rangle$, $\langle\!\langle \zeta, \eta\rangle\!\rangle = 1 = \langle\!\langle \xi, \xi\rangle\!\rangle$ and $\langle\!\langle \xi, \zeta\rangle\!\rangle = 0 = \langle\!\langle \xi, \eta\rangle\!\rangle$. Then

$$\beta = \phi\zeta + \psi\xi,$$

where $\phi = \langle\!\langle \beta, \eta\rangle\!\rangle$ and $\psi = \langle\!\langle \beta, \xi\rangle\!\rangle$. Flatness of β implies that $\dim\ker\psi = n-1$. We claim that $\ker\psi$ is an eigenspace of A. Given $Z \in \ker\psi$ we have

$$\beta(Z, X) = \phi(Z, X)\zeta \tag{31}$$

for all $X \in T_xM$. Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the canonical basis of W and write $\zeta = \sum_{j=1}^{3} a_j e_j$. Then (31) gives

$$L(Z, X) + \frac{1}{2}(1-c)\langle Z, X\rangle = a_1\phi(Z, X)$$

and

$$L(Z, X) - \frac{1}{2}(1+c)\langle Z, X\rangle = a_2\phi(Z, X).$$

Subtracting the second of the preceding equations from the first yields

$$\langle Z, X\rangle = (a_1 - a_2)\phi(Z, X),$$

which implies that $a_1 - a_2 \neq 0$ and

$$\phi(Z, X) = \frac{1}{a_1 - a_2}\langle Z, X\rangle.$$

Moreover, we also obtain from (31) that

$$\langle AZ, X\rangle = a_3\phi(Z, X) = \frac{a_3}{a_1 - a_2}\langle Z, X\rangle,$$

which proves our claim. □

1.7 Proof of Theorem 8

In order to prove Theorem 8, first recall that a necessary and sufficient condition for a three-dimensional Riemannian manifold M^3 to be conformally flat is that its Schouten tensor L be a *Codazzi tensor*, that is,

$$(\nabla_X L)(Y, Z) = (\nabla_Y L)(X, Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where

$$(\nabla_X L)(Y, Z) = X(L(Y, Z)) - L(\nabla_X Y, Z) - L(Y, \nabla_X Z).$$

Now let $f\colon M^3 \to \mathbb{Q}_s^4(c)$ be a holonomic hypersurface whose associated pair (v, V) satisfies (4). Then $v = (v_1, v_2, v_3)$ is a null vector with respect to the Lorentzian inner product $\langle\,,\,\rangle$ given by (22), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, and $V = (V_1, V_2, V_3)$ is a unit space-like vector orthogonal to v. Thus, we may write

$$V = \frac{\rho}{v_2}v + \frac{\lambda}{v_2}(-v_3, 0, v_1), \quad \lambda = \pm 1,$$

for some $\rho \in C^\infty(M)$, which is equivalent to

$$V_1 = \frac{1}{v_2}(V_2 v_1 - \lambda v_3) \quad \text{and} \quad V_3 = \frac{1}{v_2}(V_2 v_3 + \lambda v_1). \tag{32}$$

In particular,

$$V_i v_j - V_j v_i = -\lambda v_k, \quad 1 \leq i < j \leq 3, \quad k \notin \{i, j\},$$

hence the principal curvatures $\lambda_j = \frac{V_j}{v_j}$, $1 \leq j \leq 3$, are pairwise distinct.

The eigenvalues μ_1, μ_2 and μ_3 of the Schouten tensor L are given by

$$2\mu_j = c + \epsilon(\lambda_i\lambda_j + \lambda_k\lambda_j - \lambda_i\lambda_k), \quad 1 \leq j \leq 3,$$

where λ_j, $1 \leq j \leq 3$, are the principal curvatures of f. Define

$$\phi_j = v_j(\lambda_i\lambda_j + \lambda_k\lambda_j - \lambda_i\lambda_k), \quad 1 \leq j \leq 3. \tag{33}$$

That L is a Codazzi tensor is then equivalent to the equations

$$\frac{\partial \phi_j}{\partial u_i} = h_{ij}\phi_i, \quad 1 \leq i \neq j \leq 3. \tag{34}$$

Replacing $\lambda_j = \frac{V_j}{v_j}$ in (33) and using (32) we obtain

$$\phi_1 = \frac{1}{v_2^2}(-2\lambda V_2 v_3 + \left(V_2^2 - 1)v_1\right), \quad \phi_2 = \frac{1}{v_2}\left(V_2^2 + 1\right)$$

and

$$\phi_3 = \frac{1}{v_2^2}\left((V_2^2 - 1)v_3 + 2\lambda V_2 v_1\right).$$

It is now a straightforward computation to verify (34) by using equations (i) and (iv) of system (2) together with Eqs. (5) and (6).

Conversely, assume that $f\colon M^3 \to \mathbb{Q}_s^4(c)$ is an isometric immersion with three distinct principal curvatures $\lambda_1 < \lambda_2 < \lambda_3$ of a conformally flat manifold. Let $\{e_1, e_2, e_3\}$ be a correspondent orthonormal frame of principal directions. Then $\{e_1, e_2, e_3\}$ also diagonalyzes the Schouten tensor L, and the correspondent eigenvalues are

$$2\mu_j = \epsilon\left(\lambda_i\lambda_j + \lambda_j\lambda_k - \lambda_i\lambda_k\right) + c, \quad 1 \leq j \leq 3. \tag{35}$$

The Codazzi equations for f and L are, respectively,

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\langle\nabla_{e_j}e_i, e_j\rangle, \qquad i \neq j, \tag{36}$$

$$(\lambda_j - \lambda_k)\langle\nabla_{e_i}e_j, e_k\rangle = (\lambda_i - \lambda_k)\langle\nabla_{e_j}e_i, e_k\rangle, \qquad i \neq j \neq k. \tag{37}$$

and

$$e_i(\mu_j) = (\mu_i - \mu_j)\langle \nabla_{e_j} e_i, e_j\rangle, \quad i \neq j, \tag{38}$$

$$(\mu_j - \mu_k)\langle \nabla_{e_i} e_j, e_k\rangle = (\mu_i - \mu_k)\langle \nabla_{e_j} e_i, e_k\rangle, \quad i \neq j \neq k. \tag{39}$$

Substituting (35) into (39), and using (37), we obtain

$$(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)\langle \nabla_{e_i} e_j, e_k\rangle = 0, \quad i \neq j \neq k.$$

Since λ_1, λ_2 and λ_3 are pairwise distinct, it follows that

$$\langle \nabla_{e_i} e_j, e_k\rangle = 0, \quad i \neq j \neq k \neq i. \tag{40}$$

Differentiating (35) with respect to e_i, we obtain

$$2e_i(\mu_j) = \epsilon\left[(\lambda_i + \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_j - \lambda_i)e_i(\lambda_k)\right]. \tag{41}$$

On the other hand, it follows from (35), (36) and (38) that

$$e_i(\mu_j) = \epsilon\lambda_k e_i(\lambda_j). \tag{42}$$

Hence

$$(\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_i - \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_i)e_i(\lambda_k) = 0. \tag{43}$$

Now let $\{\omega_1, \omega_2, \omega_3\}$ be the dual frame of $\{e_1, e_2, e_3\}$ and define the one-forms γ_j, $1 \leq j \leq 3$, by

$$\gamma_j = x_j\omega_j, \quad x_j = \sqrt{\delta_j(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}, \quad 1 \leq j \neq i \neq k \neq j \leq 3, \tag{44}$$

where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. As in the proof of Theorem 6, one can check that (40) and (43) are precisely the conditions for the one-forms γ_j, $1 \leq j \leq 3$, to be closed.

Therefore, each point $x \in M^3$ has an open neighborhood V where one can find functions $u_j \in C^\infty(V)$, $1 \leq j \leq 3$, such that $du_j = \gamma_j$, and we can choose V so small that $\Phi = (u_1, u_2, u_3)$ is a diffeomorphism of V onto an open subset $U \subset \mathbb{R}^3$, that is, (u_1, u_2, u_3) are local coordinates on V. From $\delta_{ij} = du_j(\partial_i) = x_j\omega_j(\partial_i)$ it follows that $\partial_j = v_j e_j$, $1 \leq j \leq 3$, with $v_j = x_j^{-1}$. Now notice that

$$\sum_{j=1}^{3} \delta_j v_j^2 = \sum_{i,k\neq j=1}^{3} \frac{1}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)} = 0,$$

$$\sum_{j=1}^{3} \delta_j v_j V_j = \sum_{j=1}^{3} \delta_j \lambda_j v_j^2 = \sum_{i,k\neq j=1}^{3} \frac{\lambda_j}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)} = 0$$

and

$$\sum_{j=1}^{3} \delta_j V_j^2 = \sum_{j=1}^{3} \delta_j \lambda_j^2 v_j^2 = \sum_{i,k\neq j=1}^{3} \frac{\lambda_j^2}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)} = 1.$$

It follows that (v, V) satisfies (4). $\square$

1.8 Proof of Proposition 9

By Theorem 8, f is locally a holonomic hypersurface whose associated pair (v, V) is given in terms of the principal curvatures $\lambda_1 < \lambda_2 < \lambda_3$ of f by

$$v_j = \sqrt{\frac{\delta_j}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}} \quad \text{and} \quad V_j = \lambda_j v_j, \quad 1 \le j \le 3, \tag{45}$$

where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. Moreover, we have seen in the proofs of Theorems 6 and 8, respectively, that λ_1, λ_2 and λ_3 satisfy (30) and (43). It is easily checked that (43) is equivalent to

$$(\lambda_k - \lambda_i)e_i(\lambda_i\lambda_j) = (\lambda_j - \lambda_i)e_i(\lambda_i\lambda_k), \quad 1 \le i \ne j \ne k \ne i \le 3,$$

whereas multiplying (43) by C and adding (30) gives

$$\lambda_k(\lambda_k - \lambda_i)e_i(\lambda_i\lambda_j) = \lambda_j(\lambda_j - \lambda_i)e_i(\lambda_i\lambda_k), \quad 1 \le i \ne j \ne k \ne i \le 3.$$

Since λ_1, λ_2 and λ_3 are pairwise distinct, the two preceding equations together imply that

$$e_i(\lambda_i\lambda_j) = 0, \quad 1 \le i \ne j \le 3.$$

Assuming that $\lambda_j \ne 0$ for $1 \le j \le 3$, we can write

$$\lambda_i\lambda_j = \iota_k\phi_k^2, \quad \iota_k \in \{-1, 1\}, \quad 1 \le i \ne j \ne k \ne i \le 3, \tag{46}$$

for some positive smooth functions $\phi_k = \phi_k(u_k)$, $1 \le k \le 3$. It follows from (46) that

$$\lambda_j = \epsilon_j \frac{\phi_i\phi_k}{\phi_j}, \tag{47}$$

where $\epsilon_j = \frac{\lambda_j}{|\lambda_j|}$, $1 \le j \le 3$. Since $\lambda_1 < \lambda_2 < \lambda_3$ we have

$$\epsilon_k\phi_i^2 - \epsilon_i\phi_k^2 > 0, \quad 1 \le i < k \le 3.$$

Substituting (47) into (45), we obtain that

$$v_j = \frac{\phi_j}{\psi_i\psi_k}, \quad 1 \le j \le 3, \tag{48}$$

where $\psi_j = \sqrt{\epsilon_k\phi_i^2 - \epsilon_i\phi_k^2}$, and

$$V_j = \lambda_j v_j = \epsilon_j \frac{\phi_i\phi_k}{\psi_i\psi_k}, \quad i, k \ne j, \quad i < k.$$

We obtain from (48) that

$$h_{ij} = \frac{1}{v_j}\frac{\partial v_j}{\partial u_i} = \frac{\psi_i\psi_k}{\phi_j}\frac{\phi_j}{\psi_i\psi_k^2}\left(-\frac{\partial \psi_k}{\partial u_i}\right) = -\frac{1}{\psi_k}\frac{\partial \psi_k}{\partial u_i}. \tag{49}$$

On the other hand, equation (iv) of system (2) yields

$$h_{ij} = \frac{1}{V_j}\frac{\partial V_j}{\partial u_i} = \frac{\psi_i\psi_k}{\phi_i\phi_k}\frac{\phi_k}{\psi_i\psi_k^2}\left(\frac{d\phi_i}{du_i}\psi_k - \phi_i\frac{\partial \psi_k}{\partial u_i}\right) = \frac{1}{\phi_i}\frac{d\phi_i}{du_i} - \frac{1}{\psi_k}\frac{\partial \psi_k}{\partial u_i}. \tag{50}$$

Comparing (49) and (50), we obtain that

$$\frac{d\phi_i}{du_i} = 0, \quad 1 \le i \le 3.$$

This implies that $\frac{\partial \psi_k}{\partial u_i} = 0$ for all $1 \le i \ne k \le 3$, and hence $h_{ij} = 0$ for all $1 \le i \ne j \le 3$. But then equation (ii) of system (2) gives

$$\epsilon_s \lambda_i \lambda_j + c = 0$$

for all $1 \le i \ne j \le 3$, which implies that $-\epsilon_s c > 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \sqrt{-\epsilon_s c}$, a contradiction. Thus, one of the principal curvatures must be zero, and the result follows from part $b)$ of Theorem 4. □

1.9 Proof of Proposition 10

Before proving Proposition 10, given a hypersurface $f : M^3 \to \mathbb{Q}_s^4(c)$ we compute the pair (v^t, V^t) associated to a parallel hypersurface $f_t : M^3 \to \mathbb{Q}_s^4(c) \subset \mathbb{R}^5_{s+\epsilon_0}$ to f, with $\epsilon_0 = 0$ or 1, corresponding to $c > 0$ or $c < 0$, respectively.

Set $\epsilon_c = c/|c|$ and $\check{\epsilon} = \epsilon_s \epsilon_c$. Let φ and ψ be defined by

$$(\varphi(t), \psi(t)) = \begin{cases} (\cos(\sqrt{|c|}t), \sin(\sqrt{|c|}t)), & \text{if} \quad \epsilon = 1, \\ (\cosh(\sqrt{|c|}t), \sinh(\sqrt{|c|}t)), & \text{if} \quad \epsilon = -1. \end{cases}$$

If N is one of the unit normal vector fields to f and $i : \mathbb{Q}_s^4(c) \to \mathbb{R}^5_{s+\epsilon_0}$ is the inclusion, then

$$i \circ f_t = \varphi(t) i \circ f + \frac{\psi(t)}{\sqrt{|c|}} i_* N.$$

We denote by M_t^3 the manifold M^3 endowed with the metric induced by f_t.

Lemma 13 *Let $f : M^3 \to \mathbb{Q}_s^4(c)$ be a holonomic hypersurface. Then any parallel hypersurface $f_t : M_t^3 \to \mathbb{Q}_s^4(c)$ to f is also holonomic and the pairs (v, V) and (v^t, V^t) associated to f and f_t, respectively, are related by*

$$\begin{cases} v_i^t = \varphi(t) v_i - \frac{\psi(t)}{\sqrt{|c|}} V_i \\ V_i^t = \check{\epsilon} \sqrt{|c|} \psi(t) v_i + \varphi(t) V_i. \end{cases} \tag{51}$$

In particular, $h_{ij}^t = h_{ij}$.

Proof We have

$$f_{t*} = \varphi(t) f_* + \frac{\psi(t)}{\sqrt{|c|}} N_* = f_* \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right), \tag{52}$$

thus a unit normal vector field to f_t is $N_t = -\check{\epsilon} \sqrt{|c|} \psi(t) f + \varphi(t) N$, and

$$\begin{aligned} N_{t*} &= f_* \left(-\check{\epsilon} \sqrt{|c|} \psi(t) I - \varphi(t) A \right) \\ &= -f_{t*} \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right)^{-1} \left(\check{\epsilon} \sqrt{|c|} \psi(t) I + \varphi(t) A \right). \end{aligned}$$

which implies that

$$A_t = \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right)^{-1} \left(\check{\epsilon} \sqrt{|c|} \psi(t) I + \varphi(t) A \right). \tag{53}$$

It follows from (52) and (53) that $\tilde{f}$ is also holonomic with associated pair given by (51). The assertion on h^t_{ij} follows from a straightforward computation.

Proof of Proposition 10: In view of (51), conditions (3) for (v^t, V^t) (with $\tilde{c} = 0$) follow immediately from those for (v, V). □

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