

A characteristic-free criterion of birationality

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Abstract

One develops *ab initio* the theory of rational/birational maps over reduced, but not necessarily irreducible, projective varieties in arbitrary characteristic. A virtual numerical invariant of a rational map is introduced, called the Jacobian dual rank. It is proved that a rational map in this general setup is birational if and only if the Jacobian dual rank is well defined and attains its maximal possible value. Even in the “classical” case where the source variety is irreducible there is some gain for this invariant over the degree of the map because, on one hand, it relates naturally to constructs in commutative algebra and, on the other hand, is effectively computable. Applications are given to results only known so far in characteristic zero. One curious byproduct is an alternative approach to deal with the result of Dolgachev concerning the degree of a plane polar Cremona map.

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1. Introduction

A rational map $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ between projective spaces is defined by $m + 1$ forms $\mathbf{f} = \{f_0, \dots, f_m\}$ of the same degree in $n + 1$ variables not all vanishing. The idea of looking at the syzygies of the forms $\mathbf{f}$ goes back to the work of Hulek, Katz and Schreyer [6] in the case where $m = n$, as a method of obtaining sufficient conditions for $\mathfrak{F}$ to be a Cremona map. In [8] this method has been developed to a greater extent when $m \geq n$. Still more recently, the present third author pushed the method further to the study of general rational maps between two integral projective schemes in arbitrary characteristic by an extended ideal-theoretic method emphasizing the role of the Rees algebra associated to the ideal generated by $\mathbf{f}$ [9].

Although the general results were characteristic-free, some particular questions remained a challenge in positive characteristic (but see [16] for the case of rational maps defined by monomials). Even in characteristic zero a few issues were left answered in these papers.

The goal of this work is both a generalization and a great simplification of the basic theory of rational maps in arbitrary characteristic. We consider rational maps defined on reduced, but not necessarily, irreducible projective varieties. Though usually thought as a trivial extension from the case where the source is integral, to establish it on firm ground requires non-trivial proofs on several passages. Moreover, a *bona fide* generalization ought to work well through the various versions of the notion of birationality. In order to proceed in a rigorous way we found it essential to define everything *ab initio*, so to avoid often inevitable hand waving from the integrality habits.

The simplification comes about by showing that birationality is controlled by the behavior of a unique (virtual) numerical invariant – called the *Jacobian dual rank* of a rational map. “Virtual” refers to the fact that it may not be well defined in general – as is the case of the rank of a matrix over an arbitrary reduced ring. Nevertheless, it will be well defined if the rational map is birational onto its image no matter how badly behaved is the target of the map.

True enough, the classical theory also depends only on the degree of the rational map. However, the latter is in full control only in the integral case. Moreover, in positive characteristic one treats the inseparability degree apart from the main stream of the natural ideas in birational theory. The new invariant introduced here looks more intrinsic and makes no explicit reference to inseparability. The hidden reason is that the Jacobian dual rank appeals to just a subset of the defining equations of a suitable blow-up which are linear in the target variables, thus bypassing the characteristic effect. Accordingly, all results throughout the paper, including the main criterion and the applications, will be characteristic-free.

Finally, the Jacobian dual rank is straightforwardly effectively computable in the usual implementation of the Gröbner basis algorithm, an appreciable advantage over the field degree. In addition, the invariant calls attention to several aspects of the theory of Rees algebras and base ideals of maps, a trend sufficiently shown in many modern accounts (see, e.g., [2,5–7]).

The paper is divided in two sections.

The first section hinges on the needed background to state the general criterion of birationality. In the initial subsections we develop the ground material on rational and birational maps on a reduced source. Our approach is entirely algebraic, but we mention the transcription to the geometric side. A degree of care is required to show that the present notion is stable under the expected manipulations from the “classical” case. One main result in this part is Proposition 2.11 which drives us back to an analogue of the field extension version.

The main core is the subsequent subsection, where we introduce the Jacobian dual rank and prove the basic characteristic-free criterion of birationality in terms of this rank. The criterion also holds component-wise as possibly predictable (but not obviously proved!). We took pains

to transcribe the criterion into purely geometric terms, except for the Jacobian dual rank itself, whose geometric meaning is not entirely apparent at this stage. This concept has evolved continuously from previous notions, of which the cradle is the Jacobian dual fibration of [10] (see the subsequent [8] and [9]). The base ideal of a rational map will play an essential role often not obviously foreseen in the basic definitions.

Among the techniques for this part, a substantial role is played by the Rees algebra of an ideal in a more vivid context. Thus, let $\mathbf{f} = \{f_0, \dots, f_m\} \subset R$ be an ordered set of forms of the same degree, where R denotes a standard graded k -algebra, with k a field and $\dim R \geq 1$. Thus, one has a presentation $R \simeq k[x_0, \dots, x_n] = k[\mathbf{X}]/\mathfrak{a}$ with $k[\mathbf{X}] = k[X_0, \dots, X_n]$ standard graded and $\mathfrak{a}$ a homogeneous ideal. Let $R[\mathbf{Y}] = R[Y_0, \dots, Y_m]$ be a polynomial ring over R with the standard bigrading where $\deg(x_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$. Since $\mathbf{f}$ is generated in a fixed degree, the Rees algebra

$$\mathcal{R}_R((\mathbf{f})) := R \oplus I \oplus I^2 \oplus \dots \simeq R[It] \subset R[t]$$

is a standard bigraded k -algebra. Mapping $Y_j \mapsto f_j t$ yields a presentation $R[\mathbf{Y}]/\mathcal{J} \simeq \mathcal{R}_R((\mathbf{f}))$, with $\mathcal{J}$ a bihomogeneous *presentation ideal*. It will be seen, among the preliminaries, that $\mathcal{J}$ depends only on the rational map defined by $\mathbf{f}$ and not on this particular representative. Write

$$\mathcal{J} = \bigoplus_{(p,q) \in \mathbb{N}^2} \mathcal{J}_{(p,q)},$$

where $\mathcal{J}_{(p,q)}$ denotes the k -vector space of forms of bidegree (p, q) .

One first relevant bihomogeneous piece for our purpose is $\mathcal{J}_{0,*}$, which is spanned by the forms of bidegree $(0, q)$, for all $q \geq 1$. Since these forms have coefficients only in k , one can view them as elements of $k[\mathbf{Y}]$. As such they generate the ideal of all polynomial relations (over k) of $\mathbf{f}$. Thus,

$$k[\mathbf{Y}]/(\mathcal{J}_{0,*})k[\mathbf{Y}] \simeq k[\mathbf{f}],$$

as standard graded k -algebras, after a degree renormalization in $k[\mathbf{f}]$.

For birationality, yet another bihomogeneous piece is as important:

$$\mathcal{J}_{1,*} := \bigoplus_{r \in \mathbb{N}} \mathcal{J}_{1,q}$$

with $\mathcal{J}_{1,q}$ denoting the bigraded piece of $\mathcal{J}$ spanned by the forms of bidegree $(1, q)$ for all $q \geq 0$. Note that $\mathcal{J}_{1,*} = \{0\}$ may very well be the case in general. The definition of the Jacobian dual rank is drawn from this datum in form of the rank of a Jacobian-like matrix of a suitable generating set of such forms. Although this rank may not be well defined for an arbitrary rational map, it is stable regarding the various notions introduced (see, e.g., Lemma 2.13). Moreover, the main result (Theorem 2.18) says that *the rational map with source $\text{Proj}(R)$ defined by $\mathbf{f}$ is birational onto its image if and only if the Jacobian dual rank is well defined and attains its maximal possible value*.

The section develops a good deal of preliminary propositions, of which Proposition 2.15 deserves special notice. Due to the technicality involved in the concepts we refer to the appropriate places in the section.

Quite a bit of deep classical theory seems to be embedded into the above two graded pieces of the defining equations of the Rees algebra. The purpose of the Jacobian dual rank is roughly to link between these pieces. While it is not entirely clear how they entangle, the application in this work surfaces when this rank acquires maximal possible value.

The second section is concerned with applications of the main criterion.

Since the criterion is stated in terms of an effective numerical invariant, the obvious applicability is to finding out whether a rational map given explicitly by a representative is birational and, in the affirmative case, to exhibiting a representative of the inverse map. However, here one has in mind applications of a different nature. Thus, we have collected a few results previously known in characteristic zero, now fully available in arbitrary characteristic. The flagship is Theorem 3.2. One of its consequences is a result for plane Cremona maps, saying that the only such maps whose base ideal is locally a complete intersection at its minimal primes are the classical quadratic maps (up to a projective change of coordinates). In particular, the result of Dolgachev [3] is equivalent to showing that the gradient ideal of a homogeneous homaloidal polynomial in 3 variables is locally a complete intersection at its minimal primes.

We close the section by introducing yet another numerical invariant, called the semilinear generation degree. It would be interesting to capture its full meaning in the present and related theories – such as elimination.

2. A criterion of birationality

2.1. Prolegomenon: the Koszul–Hilbert lemma

The following curious piece of linear algebra, for which we have found no explicit reference in the literature, will be useful in the subsequent parts of this section.

Proposition 2.1. *Let A be a commutative ring and n a positive integer. Let ρ stand for an $(n+1) \times n$ matrix with arbitrary entries in A and let $\Delta = \{\Delta_0, \dots, \Delta_n\}$ stand for its (ordered, signed) n -minors. Then the Koszul relations of Δ belong to the submodule*

$$\text{Im}(\rho) \subset A^{n+1}.$$

Proof. Let $B = A[\mathbf{X}]$ be a polynomial ring in $(n+1) \times n$ indeterminates and let $\mathcal{X}$ denote the corresponding $(n+1) \times n$ matrix whose entries are $\mathbf{X}$.

We need:

Lemma 2.2. *Let D denote a Noetherian factorial subdomain of A – e.g., the prime subring of A . Then the ideal of $D[\mathbf{X}]$ generated by the n -minors of $\mathcal{X}$ has grade ≥ 2 .*

Proof. This is clear as, say, $\Delta_1 \in D[\mathbf{X}]$ generates a prime ideal and, say, Δ_2 is not a multiple of Δ_1 . $\square$

As a consequence, the well-known Hilbert–Burch principle says that the sequence

$$0 \rightarrow D[\mathbf{X}]^n \xrightarrow{\mathcal{X}} D[\mathbf{X}]^{n+1} \xrightarrow{\Delta} D[\mathbf{X}]$$

is exact. Clearly, then the Koszul relations of Δ belong to the image of the map $\mathcal{X}$ as this is exactly the matrix of first syzygies of Δ .

Now specialize the variables $\mathbf{X}$ to the entries of ρ by means of the obvious homomorphism $D[\mathbf{X}] \rightarrow A$ extending the inclusion $D \subset A$. Since the formation of determinants and the Koszul complex specialize, it is clear that the specialization preserves the above relation. $\square$

Remark 2.3. Note that one is not asserting that ρ is the entire matrix of syzygies of Δ , which is only true if the grade of Δ is at least 2 as well.

2.2. Preliminaries on rational maps on reduced varieties

We first fix our standing setup and notation.

Let k denote a fixed, but otherwise arbitrary, ground field. As usual, in order that a projective set of zeros have enough points one allows coordinates to lie on an algebraically closed field containing k .

Let $R = k[\mathbf{x}] = k[x_0, \dots, x_n] \simeq k[\mathbf{X}]/\mathfrak{a} = k[X_0, \dots, X_n]/\mathfrak{a}$ denote a standard graded k -algebra over a field k of positive dimension. In the sequel we always write $\mathbf{X}$ and $\mathbf{x}$ for $X_0, \dots, X_n$ and $x_0, \dots, x_n$, respectively. The same notation principle will apply to other sets of variables as needed.

Rational data and rational maps

Let $K(R)$ denote the homogeneous total ring of quotients of R – i.e., the ring of fractions $\mathfrak{M}^{-1}R$, where $\mathfrak{M}$ is the set of homogeneous regular elements of R not belonging to any of the minimal primes of R .

We will be dealing with ordered $(m+1)$ -sets $\mathbf{f} = \{f_0, \dots, f_m\}$ of forms of the same degree in R . Let us say that two such ordered $(m+1)$ -sets $\mathbf{f} = \{f_0, \dots, f_m\}$ and $\mathbf{f}' = \{f'_0, \dots, f'_m\}$, are EQUIVALENT if the corresponding tuples $(f_0, \dots, f_m)$ and $(f'_0, \dots, f'_m)$ are proportional by means of a homogeneous invertible element of $K(R)$. This means that there exist regular homogeneous elements $F, F' \in R$ such that $Ff'_j = F'f_j$, for $j = 0, \dots, m$. This easily seen to be an equivalence relation.

Note that equivalent ordered homogeneous $(m+1)$ -sets $\mathbf{f}, \mathbf{f}' \subset R$ induce a graded k -isomorphism of the respective representative algebras $k[\mathbf{f}]$ and $k[\mathbf{f}']$, such that $f_j \leftrightarrow f'_j$.

Definition 2.4. A RATIONAL $(m+1)$ -DATUM OF DEGREE d on R is an ordered set of $m+1$ forms $\mathbf{f} = \{f_0, \dots, f_m\} \subset R$ of the same degree ≥ 1 satisfying the following conditions:

- (i) R is torsionfree over $k[\mathbf{f}]$ via the ring extension $k[\mathbf{f}] \subset R$.
- (ii) The ideal $(\mathbf{f})R \subset R$ is not contained in any minimal prime of R .

Note that the first condition reads in the present circumstances that every minimal prime of R contracts to a minimal prime of $k[\mathbf{f}]$, while the second says that the ideal $(\mathbf{f}) \subset R$ has a regular element.

An m -TARGETED RATIONAL MAP with SOURCE $\text{Proj}(R) \subset \mathbb{P}_k^m$ is a collection of equivalent rational $(m+1)$ -data.

Lemma 2.5. If k is an infinite field, two homogeneous $(m+1)$ -sets $\mathbf{f}, \mathbf{f}' \subset R$ satisfying condition (ii) of the above definition are equivalent if and only if the matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_m \\ f'_0 & f'_1 & \cdots & f'_m \end{pmatrix}$$

has rank 1.

Proof. Indeed, the rank condition means that the 2×2 minors vanish (which is readily implied by equivalence as defined) and that the ideal generated by the entries has a regular element (which is obviously the case if one of the ideals generated by the rows has a regular element). To see the converse, since k is infinite one can choose suitable $\lambda_j \in k$ such that both $\lambda_0 f_0 + \cdots + \lambda_m f_m$ and $\lambda_0 f'_0 + \cdots + \lambda_m f'_m$ are regular elements. By hypothesis, $f_j f'_i = f_i f'_j$ for all pairs i, j . Therefore, for arbitrary i , one has $F' f_i = F f'_i$, where $F = \lambda_0 f_0 + \cdots + \lambda_m f_m$ and $F' = \lambda_0 f'_0 + \cdots + \lambda_m f'_m$. $\square$

The rank property was taken in [9] as definition of equivalent representatives of the same rational map.

Any individual rational $(m+1)$ -datum is said to be a REPRESENTATIVE of the map and, in an informal speech, we say that it defines the rational map. Throughout we denote a rational map by $\mathfrak{F} : \mathbf{f}$, where $\mathbf{f}$ is a particular representative. Often we identify the map with one representative, which will cause no confusion. As remarked above, equivalent rational $(m+1)$ -data $\mathbf{f}, \mathbf{f}' \subset R$ induce a k -isomorphism of the respective representative algebras $k[\mathbf{f}]$ and $k[\mathbf{f}']$, such that $f_j \leftrightarrow f'_j$. Although this isomorphism is graded, it is not homogeneous (of degree 0). Nevertheless, one can renormalize the gradings of all these algebras so they become standard k -algebras. What this actually means is that one can take a common polynomial representation $k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}]$ for all the representatives of the same m -targeted rational map, with $k[\mathbf{Y}]/\mathfrak{b}$ standard.

Base ideal and stability

The ideal $(\mathbf{f}) \subset R$ generated by a representative $\mathbf{f} \subset R$ is called a BASE IDEAL of the map. At the level of base ideals, two representatives $\mathbf{f}$ and $\mathbf{f}'$ of the same map give an isomorphism of $(\mathbf{f})$ and $(\mathbf{f}')$ as R -submodules of $K(R)$, i.e., as fractional ideals.

The notion of a representative or of a base ideal is pretty stable in the following sense.

Lemma 2.6. *Let $\mathbf{f} = \{f_0, \dots, f_m\}$ and $\mathbf{f}' = \{f'_0, \dots, f'_m\}$ be equivalent ordered sets in R . Then $\mathbf{f}$ is a rational datum if and only if $\mathbf{f}'$ is a rational datum.*

Proof. By the obvious symmetry in an argument, it suffices to show one implication. Thus, assume that $\mathbf{f}$ satisfies the conditions stated in Definition 2.4. As noted above, $k[\mathbf{f}] \simeq k[\mathbf{f}']$, so there is a bijection between the respective sets of minimal primes. This shows that condition (i) of Definition 2.4 holds for $\mathbf{f}'$. Since the k -isomorphism induces an isomorphism of the ideals $(\mathbf{f}), (\mathbf{f}') \subset R$ as R -modules (also remarked before), condition (ii) follows as well. $\square$

Criterion in terms of total rings of quotients

The usual “field” criterion reads as well. Namely, choose a fixed presentation $S = k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}]$ for an arbitrary algebra representative of degree d of an m -targeted rational map. There are actually three versions of the criterion, as follows. Since $S \hookrightarrow R$ is torsionfree, it induces the following maps:

- (1) An injective graded homomorphism $K(S) \hookrightarrow K(R)$ of degree $d-1$ of the respective homogeneous total rings of quotients.

- (2) An injective *homogeneous* homomorphism (i.e., a graded homomorphism of degree zero) $K(S) \hookrightarrow K(R^{(d)})$, where $R^{(d)} \subset R$ is the d th Veronese subring of R .
- (3) An injective (non-graded) homomorphism $K(S)_0 \hookrightarrow K(R)_0$ in degree zero.

Note that, by definition, $K(R^{(d)})_0 = K(R)_0$, so passing to the Veronese in degree zero is tantamount to doing it “on the nose” over R .

Composition

Two such objects can be “composed” in the following sense:

Lemma 2.7. *Let $R = k[\mathbf{X}]/\mathfrak{a}$ be a reduced standard graded k -algebra and let $\mathfrak{F} : \mathbf{f} \subset R$ be a rational $(m+1)$ -datum of degree d on R . Take a presentation $S := k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}]$, with $\mathbf{Y} = \{y_0, \dots, y_m\}$ variables over k . Let $\mathbf{g} := \{g_0, \dots, g_r\} \subset S$ denote a rational $(r+1)$ -datum of degree e on S . Then:*

- (i) $\mathbf{g}(\mathbf{f}) := \{g_0(\mathbf{f}), \dots, g_r(\mathbf{f})\}$ is a rational $(r+1)$ -datum of degree de on R .
- (ii) If $\mathbf{f}' \subset R$ and $\mathbf{g}' \subset S$ are representatives of $\mathbf{f}$ and $\mathbf{g}$, respectively, then $\mathbf{g}'(\mathbf{f}') := \{g'_0(\mathbf{f}'), \dots, g'_r(\mathbf{f}')\}$ is a representative of the rational map defined by $\mathbf{g}(\mathbf{f})$.

Proof. (i) By assumption, R is torsionfree over $k[\mathbf{f}] \simeq S$ and S is torsionfree over $k[\mathbf{g}]$. Therefore, R is torsionfree over $k[\mathbf{g}]$. Likewise, by assumption, $(\mathbf{g})S$ is not contained in any minimal prime of S , i.e., its image $\mathbf{g}(\mathbf{f})k[\mathbf{f}]$ via the isomorphism $S \simeq k[\mathbf{f}]$ is not contained in any minimal prime of $k[\mathbf{f}]$. Therefore, by torsionfreeness of R over $k[\mathbf{f}]$, the ideal $(\mathbf{g}(\mathbf{f}))R$ is not contained in any minimal prime of R .

(ii) Let $S = k[\mathbf{Y}]/\mathfrak{b}$ be a common presentation of the algebras $k[\mathbf{f}]$ and $k[\mathbf{f}']$. By assumption, there exist $F, F' \in R$ homogeneous such that $Ff'_j = F'f_j$, for $j = 0, \dots, m$ and, likewise, $G = G(\mathbf{y})$, $G' = G'(\mathbf{y}) \subset S$ homogeneous such that $G(\mathbf{y})g'_\ell(\mathbf{y}) = G'(\mathbf{y})g_\ell(\mathbf{y})$, for $\ell = 0, \dots, r$. We have appended $\mathbf{y}$ to the latter to emphasize that they are elements of $S = k[\mathbf{Y}]/\mathfrak{b}$. Using the isomorphism $k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}']$, for instance, one has

$$G(\mathbf{f}')g'_\ell(\mathbf{f}') = G'(\mathbf{f}')g_\ell(\mathbf{f}'). \quad (1)$$

Substituting for $f'_j = (F'/F)f_j$, $j = 0, \dots, m$ in $g_\ell(\mathbf{f}')$ and pulling out the obvious factor yields $F^e \cdot g_\ell(\mathbf{f}') = F'^e \cdot g_\ell(\mathbf{f})$, where e stands for the common degrees of the g_ℓ 's. Multiplying by F^e both members of (1) and bringing into it the last equality gives

$$(F^e G(\mathbf{f}'))g'_\ell(\mathbf{f}') = G'(\mathbf{f}')(F^e g_\ell(\mathbf{f}')) = (G'(\mathbf{f}')F'^e)g_\ell(\mathbf{f}),$$

for $\ell = 0, \dots, r$. Now, $G(\mathbf{f}')$ and $G'(\mathbf{f}')$ are regular (homogeneous) elements of $k[\mathbf{f}]$ and $k[\mathbf{f}']$, respectively as they are both images of regular (homogeneous) elements of $S = k[\mathbf{Y}]/\mathfrak{b}$ under a graded k -isomorphism of algebras. Since R is torsionfree over $k[\mathbf{f}']$ they are still regular (homogeneous) elements of R . Therefore, $F^e G(\mathbf{f}')$ and $G'(\mathbf{f}')F'^e$ are regular homogeneous elements of R and we are done.

ADDENDUM. As a slight control, if d (respectively, d') and e (respectively, e') are the respective degrees of f_j (respectively, f'_j) and of g_ℓ (respectively, g'_ℓ) and, likewise D (respectively, D') and E (respectively, E') are the respective degrees of F (respectively, F') and of G (respectively, G'), then

$$\begin{aligned}
\deg(G'(\mathbf{f}')F'^e) + \deg(g_\ell(\mathbf{f})) &= d'E' + eD' + de \\
&= d'(E + e' - e) + e(D + d' - d) + de \\
&= eD + d'E + d'e' \\
&= \deg(F^e G(\mathbf{f}')) + \deg(g'_\ell(\mathbf{f}')). \quad \square
\end{aligned}$$

Image of a rational map

We have seen that all representatives of the same m -targeted rational map have a common presentation $S = k[\mathbf{Y}]/\mathbf{b}$, a standard graded reduced k -algebra. Borrowing from the geometric terminology, we call $\text{Proj}(S) \subset \mathbb{P}_k^m$ the **IMAGE** of this map.

Restriction

One can introduce an additional notion, that of a restriction of an m -targeted rational map $\mathfrak{F} : \mathbf{f} \subset R$. Given a radical ideal $I \subset R$ such that $(\mathbf{f})$ is not contained in any minimal prime of R/I on R and such that R/I is torsionfree over $k[\mathbf{f}]/I \cap k[\mathbf{f}]$, then we say that the map **RESTRICTS** to $\text{Proj}(R/I)$. Thus, being able to restrict a rational map requires quite a bit – note that, if I is a prime ideal, then torsionfreeness of $k[\mathbf{f}]/I \cap k[\mathbf{f}] \subset R/I$ is trivially satisfied, so we just have to worry about the condition on the relative minimal primes, which is unavoidable in the classical geometric setup.

There is one easy situation where restriction works as expected.

Lemma 2.8. *Let $\mathfrak{F} : \mathbf{f} \subset R$ be an m -targeted rational map with source $\text{Proj}(R)$.*

- (1) *If $P \subset R$ is a minimal prime of R then the map restricts to $\text{Proj}(R/P)$ and the image of the restriction is $\text{Proj}(k[\mathbf{f}]/P \cap k[\mathbf{f}])$.*
- (2) *Conversely, if $\wp \subset k[\mathbf{f}]$ is a minimal prime, then $\text{Proj}(k[\mathbf{f}]/\wp)$ is the image of the restriction of $\mathbf{f}$ to R/P for some minimal prime $P \subset R$.*

Proof. (1) This follows immediately from the properties of the rational map.

(2) It suffices to show that $\wp$ is contracted from some prime $\mathcal{P} \subset R_+$, as in this case a minimal prime of R contained in $\mathcal{P}$ will contract to $\wp$ as well. For that, it suffices to show that $(\wp)R$ is not an irrelevant ideal in R . But if that were the case, one would have $\wp = (\mathbf{f})k[\mathbf{f}]$ which is impossible since $k[\mathbf{f}]$ is a reduced ring of positive dimension (e.g., because $(\mathbf{f})$ has a regular element contained in $k[\mathbf{f}]$ – see Remark 2.5). $\square$

What the previous lemma does for us is to allow many arguments to be reduced to the case where R is an integral domain. Of course, this has to be used judiciously.

Remark 2.9 (*Geometric transcription*). Suppose that k is algebraically closed. By a version of the projective Nullstellensatz, (2) implies that for every $P \in \text{Min}(R)$ there exists some homogeneous *submaximal* ideal (i.e., generated by $n - 1$ linearly independent 1-forms in $\mathbf{x}$) $\mathfrak{n} \subset R$ containing P but not $(\mathbf{f})$. For such an $\mathfrak{n}$, the contraction $\mathfrak{n} \cap k[\mathbf{f}] \subset k[\mathbf{f}]$ is an ideal generated by $m - 1$ linearly independent 1-forms in $\mathbf{f}$. That is to say, if we write $k[\mathbf{Y}]/\mathbf{b} \simeq k[\mathbf{f}]$, then $\mathfrak{n} \cap k[\mathbf{f}]$ is the residual image of a homogeneous submaximal ideal in $k[\mathbf{Y}]$. Geometrically, this implies that $f_0(\mathbf{a}), \dots, f_m(\mathbf{a})$ makes sense as a point in $\mathbb{P}^m$ for at least one point $\mathbf{a}$ on any given irreducible component of the projective subset $V \subset \mathbb{P}^n$ whose homogeneous coordinate ring is R . Therefore, on every such component there is a nonempty open subset where $\mathbf{f}$ affords a well-defined map.

In particular, the so obtained map $\mathfrak{F}$ is well defined on any open set $\mathcal{U} \subset V$ that intercepts non-emptily every irreducible component. The image of $\mathfrak{F}$ is the closure $W \subset \mathbb{P}^m$ of the set of images of points from any such $\mathcal{U}$. This is the projective set of $\mathbb{P}^m$ whose homogeneous coordinate ring is $k[\mathbf{Y}]/\mathfrak{b}$, i.e., after the obvious degree renormalization, $k[\mathbf{f}]$ itself.

Birational maps and “field” extension criteria

The simplest instance of an m -targeted rational map is $\mathfrak{F} : \mathbf{X} \subset R$, represented by the residues in R of the variables $\{X_0, \dots, X_n\}$ – called THE IDENTITY rational map of $\text{Proj}(R)$. The closest rational maps to the identity are the so-called birational maps. We give a formal definition.

Definition 2.10. An m -targeted rational map $\mathbf{f} \subset R = k[X_0, \dots, X_n]/\mathfrak{a}$ of degree d is said to be BIRATIONAL (ONTO THE IMAGE) if there exists an n -targeted rational map $\mathfrak{G} : \mathbf{g} \subset S = k[Y_0, \dots, Y_m]/\mathfrak{b} \simeq k[\mathbf{f}]$ such that the composite $\mathbf{g}(\mathbf{f}) \subset R$ is equivalent to the identity on $\text{Proj}(R)$.

The rational map $\mathfrak{G}$ is called an INVERSE to $\mathfrak{F}$.

The classical field theoretic criterion can be extended via the following formulation.

Proposition 2.11. Let $\mathfrak{F} : \mathbf{f} \subset R = k[X_0, \dots, X_n]/\mathfrak{a}$ denote an m -targeted rational map of degree d . The following conditions are equivalent:

- (a) $\mathfrak{F} : \mathbf{f} \subset R$ is birational onto the image.
- (b) For every minimal prime P of R , the induced m -targeted rational map

$$\mathfrak{F}|_P : \bar{\mathbf{f}} \subset \bar{R} = R/P$$

is birational onto its image and, moreover, R/P is the unique minimal component of R mapping to the latter by $\mathfrak{F}$.

- (c) The induced homomorphism in degree zero $K(S)_0 \hookrightarrow K(R)_0$ is an isomorphism.
- (d) The induced homogeneous homomorphism $K(S) \hookrightarrow K(R^{(d)})$ is an isomorphism.

Proof. The equivalence of (c) and (d) is fairly known – see, e.g., [12, Proof of Theorem 6.6] or [15, Proof of Proposition 2.1] for the argument in a more special setup. Here we trade the non-homogeneous total ring of quotients for the homogeneous one, but the reasoning is the same, namely, $K(S) = K(S)_0(f)$ and $K(R^{(d)}) = K(R^{(d)})_0(f) = K(R)_0(f)$ for some regular 1-form $f \in S$ or, up to identification $S = k[\mathbf{f}]$, a suitable k -linear combination of $f_0, \dots, f_m$. Since the maps in the statement are natural, the equivalence is clear.

Thus we are left with the mutual equivalence of (a), (b) and (c).

We note that, quite generally, if R is any reduced Noetherian ring, its ordinary total ring of quotients decomposes as the direct product of the respective fields of fractions of the domains R/P , where P runs through the minimal primes of R . If, moreover, R is standard graded over a field k (possibly even more generally) then, by restriction, this decomposition implies a similar decomposition of the *homogeneous* total ring of fractions $K(R) = \prod_{P \in \text{Min}(R)} K(R/P)$. Clearly, this decomposition induces one in degree zero as well:

$$K(R)_0 = \prod_{P \in \text{Min}(R)} K(R/P)_0, \quad (2)$$

where the factors are now fields.

(a) $\Rightarrow$ (b) This piece follows from the first statement of Lemma 2.8.

(b) $\Rightarrow$ (c) By the classical field criterion in the domain case one knows that the induced homomorphism in degree zero $K(S/P \cap S)_0 \hookrightarrow K(R/P)_0$ is an isomorphism for every minimal prime P of R . This is well known, but for the sake of completion, we repeat the argument. In fact, it is now easy to work directly with coordinates. Updating the notation, assume that $R = k[\mathbf{x}]$ is a domain. Then one has

$$K(S)_0 = k(\mathbf{y})_0 = k\left(\frac{y_1}{y_0}, \dots, \frac{y_m}{y_0}\right) \simeq k(\mathbf{f})_0 = k\left(\frac{f_1}{f_0}, \dots, \frac{f_m}{f_0}\right) \subset k(\mathbf{x})_0 = k\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

where we assume $y_0 \neq 0, x_0 \neq 0$ without loss of generality. By assumption, there are nonzero (homogeneous) $H, H' \in R$ such that $H \cdot g_\ell(\mathbf{f}) = H' \cdot x_\ell$, for $\ell = 0, \dots, n$. Then

$$\frac{x_\ell}{x_0} = \frac{g_\ell(\mathbf{f})}{g_0(\mathbf{f})} = \frac{f_0^e(g_\ell(f_1/f_0, \dots, f_m/f_0))}{f_0^e(g_0(f_1/f_0, \dots, f_m/f_0))} = \frac{g_\ell(f_1/f_0, \dots, f_m/f_0)}{g_0(f_1/f_0, \dots, f_m/f_0)}, \quad \ell = 0, \dots, n,$$

thus showing the required equality $k(f_1/f_0, \dots, f_m/f_0) = k(x_1/x_0, \dots, x_n/x_0)$.

Next, note that the map $K(S/P \cap S)_0 \hookrightarrow K(R/P)_0$ is naturally induced by the map $K(S)_0 \hookrightarrow K(R)_0$.

Then, up to identification we only need to show that if $K(S/S \cap P)_0 = K(R/P)_0$ for every $P \in \text{Min}(R)$, then $K(S)_0 = K(R)_0$. To see this, note that the hypothesis implies that, for every $\wp \in \text{Min}(S)$, exactly one prime $P \in \text{Min}(R)$ contracts to $\wp$ – i.e., the map $\text{Min}(R) \rightarrow \text{Min}(S)$ is surjective (hence, bijective by the second statement of Lemma 2.8). This is due to the fact that in the injection $K(S)_0 \hookrightarrow K(R)_0$ the decomposition (2) forces the restriction to an individual factor $K(S/\wp)$ to map diagonally to the product of the factors of $K(R/P)_0$, one for each $P \in \text{Min}(R)$ contracting to $\wp$. By the same token, this implies the sought global isomorphism $K(S)_0 \simeq K(R)_0$.

(c) $\Rightarrow$ (a) (Assuming k infinite.) Fix a regular homogeneous element $x \in (\mathbf{x}) = R_+$ of the form $x_0 + \sum_{\ell=1}^n \lambda_\ell x_\ell$, for suitable $\lambda_\ell \in k$. For $\ell = 1, \dots, n$, let $g_\ell(\mathbf{y})/g(\mathbf{y}) \in K(S)_0$ denote the (uniquely defined) inverse image of x_ℓ/x under the isomorphism $K(S)_0 \simeq K(R)_0$, for suitable homogeneous elements $g(\mathbf{y}), g_\ell(\mathbf{y}) \in (\mathbf{y}) = S_+$, with $g(\mathbf{y})$ regular. Since the isomorphism $K(S)_0 \simeq K(R)_0$ is induced by $y \mapsto f_j$, one has $g_\ell(\mathbf{f})/g(\mathbf{f}) = x_\ell/x$, for $\ell = 1, \dots, n$. Thus, $x \cdot g_\ell(\mathbf{f}) = g(\mathbf{f}) \cdot x_\ell$, for $\ell = 1, \dots, n$, with $x, g(\mathbf{f})$ regular homogeneous elements in R . Now set $g_0(\mathbf{y}) := g(\mathbf{y}) - \sum_{\ell=1}^n \lambda_\ell g_\ell(\mathbf{y})$. Then

$$\begin{aligned} g(\mathbf{f})x_0 &= g(\mathbf{f})\left(x - \sum_{\ell=1}^n \lambda_\ell x_\ell\right) = xg(\mathbf{f}) - \sum_{\ell=1}^n \lambda_\ell(g(\mathbf{f})x_\ell) = xg(\mathbf{f}) - \sum_{\ell=1}^n \lambda_\ell(xg_\ell(\mathbf{f})) \\ &= x\left(g(\mathbf{f}) - \sum_{\ell=1}^n \lambda_\ell g_\ell(\mathbf{f})\right) = x \cdot g_0(\mathbf{f}). \end{aligned}$$

Finally, note that $g_0(\mathbf{y}), \dots, g_n(\mathbf{y}) \subset S = k[\mathbf{y}]$ is a rational $(n+1)$ -datum by Lemma 2.6, hence defines an n -targeted rational map. Therefore, the rational $(n+1)$ -data $g_0(\mathbf{f}), \dots, g_n(\mathbf{f})$ and $x_0, \dots, x_n$ are equivalent, as was to be shown. $\square$

Corollary 2.12. *If an m -targeted rational map $\mathfrak{F} : \mathbf{f} \subset R = k[X_0, \dots, X_n]/\mathfrak{a}$ is birational (onto its image) with n -targeted inverse rational map $\mathfrak{G} : \mathbf{g} \subset S = k[Y_0, \dots, Y_m]/\mathfrak{b} \simeq k[\mathbf{f}]$, then the*

composite $\mathbf{f}(\mathbf{g}) \subset S$ is equivalent to the identity on $\text{Proj}(S)$. Consequently, the inverse is uniquely defined.

Proof. The first statement follows from the equivalence of (a) and (c) in the previous theorem. The second statement follows from the first in the usual way, namely writing $\equiv$ for equivalence, if $\mathbf{g}'$ is another inverse, we have

$$\mathbf{X} \equiv \mathbf{g}(\mathbf{f}) \Rightarrow \mathbf{g}' \equiv \mathbf{X}(\mathbf{g}') \equiv \mathbf{g}(\mathbf{f})(\mathbf{g}') \equiv \mathbf{g}(\mathbf{f}(\mathbf{g}')) \equiv \mathbf{g}(\mathbf{Y}) \equiv \mathbf{g}. \quad \square$$

The corollary means, as it should, that a birational map on $\text{Proj}(R)$ onto its image $\text{Proj}(S)$ has a unique “two-sided” inverse and the inverse is a birational map on $\text{Proj}(S)$ onto $\text{Proj}(R)$.

2.3. The JD-rank and its bearing to birationality

Definition and background

Let us recap the present algebraic tool for birationality that goes in parallel to the geometric notion of the linear fibers of a rational map. This concept has evolved continuously from previous notions, of which the cradle is the Jacobian dual fibration of [10] (see the subsequent [8] and [9]).

Thus, let $\mathfrak{F} : \mathbf{f} \subset R = k[\mathbf{X}]/\mathfrak{a}$ be an m -targeted rational map with source $\text{Proj}(R)$, where $k[\mathbf{X}]$ is standard graded over k , with $\dim R \geq 1$. Let $R[\mathbf{Y}] = R[Y_0, \dots, Y_m]$ be a polynomial ring over R with the standard bigrading where $\deg(x_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$. Since $\mathbf{f}$ is generated in a fixed degree, the Rees algebra

$$\mathcal{R}_R((\mathbf{f})) := R \oplus I \oplus I^2 \oplus \dots \simeq R[It] \subset R[t]$$

is a standard bigraded k -algebra. Mapping $Y_j \mapsto f_j t$ yields a presentation $R[\mathbf{Y}]/\mathcal{J} \simeq \mathcal{R}_R((\mathbf{f}))$, with $\mathcal{J}$ a bihomogeneous *presentation ideal*.

Just as the image of a rational map, $\mathcal{J}$ depends only on the rational map and not on any of its representatives. Write

$$\mathcal{J} = \bigoplus_{(p,q) \in \mathbb{N}^2} \mathcal{J}_{(p,q)},$$

where $\mathcal{J}_{(p,q)}$ denotes the k -vector space of forms of bidegree (p, q) .

First focus on $\mathcal{J}_{0,*}$, which is spanned by the forms of bidegree $(0, q)$, for all $q \geq 1$. Since these forms have coefficients only in k , one can view them as elements of $k[\mathbf{Y}]$. As such they generate the ideal of all polynomial relations (over k) of $\mathbf{f}$. Thus,

$$k[\mathbf{Y}]/(\mathcal{J}_{0,*})k[\mathbf{Y}] \simeq k[\mathbf{f}],$$

as standard graded k -algebras, after a degree renormalization in $k[\mathbf{f}]$. Thus, the ideal $(\mathcal{J}_{0,*})k[\mathbf{Y}]$ recovers the ideal $\mathfrak{b}$ of the image brought up in the previous section. We write $S = k[\mathbf{Y}]/(\mathcal{J}_{0,*})k[\mathbf{Y}] \simeq k[\mathbf{f}]$ throughout, thinking interchangeably of S as one or the other.

For birationality, the following bihomogeneous piece is no less important:

$$\mathcal{J}_{1,*} := \bigoplus_{r \in \mathbb{N}} \mathcal{J}_{1,q}$$

with $\mathcal{J}_{1,q}$ denoting the bigraded piece of $\mathcal{J}$ spanned by the forms of bidegree $(1, q)$ for all $q \geq 0$. Note that $\mathcal{J}_{1,*} = \{0\}$ may very well be the case in general.

Now, a form of bidegree $(1, *)$ can be written as $\sum_{i=0}^n Q_i(\mathbf{Y})x_i$, for suitable homogeneous $Q_i(\mathbf{Y}) \in k[\mathbf{Y}] \subset R[\mathbf{Y}]$ of the same degree. Since $\mathbf{Y}$ are indeterminates over R , two such representations of the same form imply a syzygy of $\{x_0, \dots, x_n\}$ with coefficients in k . Thus the representation is unique up to k -linear dependency of $\{x_0, \dots, x_n\}$, i.e., up to elements of $\mathfrak{a}_1$.

In particular, if the EMBEDDING DIMENSION of R – i.e., the k -vector space dimension $\text{edim}(R) := \dim_k(R_1) = n + 1 - \dim_k \mathfrak{a}_1$ – is $n + 1$ then every such form has a unique expression.

Next, one can pick a minimal set of generators of the ideal $(\mathcal{J}_{1,*})$ consisting of a finite number of forms of bidegree $(1, q)$, for various q 's. Let $\{P_1, \dots, P_s\} \subset k[\mathbf{X}, \mathbf{Y}]$ denote liftings of these bifurms and let $\{\ell_1, \dots, \ell_r\} \subset k[\mathbf{X}]$ be a k -vector space basis of $\mathfrak{a}_1$. Consider the Jacobian matrix of the polynomials $\{\ell_1, \dots, \ell_r, P_1, \dots, P_s\}$ with respect to $\mathbf{X}$, a matrix with entries in $k[\mathbf{Y}]$. Write ψ for the corresponding matrix over $S = k[\mathbf{Y}]/\mathfrak{b}$, here called the weak Jacobian matrix associated to the given set of generators of $(\mathcal{J}_{1,*})$. Note that ψ has a distinguished $(n + 1 - \text{edim}(R)) \times (n + 1)$ submatrix ρ with entries in $k[\mathbf{Y}]_0 = k$, which is non-singular. Its number of rows $r = n + 1 - \text{edim}(R)$ can be dubbed the DEGENERATION INDEX $\text{dgi}(\mathfrak{a})$ of the presentation $R \simeq k[\mathbf{X}]/\mathfrak{a}$. Clearly, this index is a lower bound for the rank of ψ over S whenever this rank is well defined.

On the other hand, by definition one clearly has $\mathbf{X} \cdot \psi^t = 0$ over $\mathcal{R}_R(\mathbf{f})$, in particular the vector $(\mathbf{X})^t$ belongs to the null space of ψ over $\mathcal{R}_R(\mathbf{f})$. If R is a domain then the rank of ψ over $\mathcal{R}_R(\mathbf{f})$ is at most n , hence for even more reason also over the residue ring $\mathcal{R}_R(\mathbf{f}) \rightarrow S$.

A consequence of Proposition 2.15 below will show that this upper bound holds true even if R is not integral, provided ψ still has a rank.

Note that a weak Jacobian matrix ψ is not uniquely defined due to the lack of uniqueness in the expression of an individual form and to the choice of bihomogeneous generators. Besides, it can be fairly “slack”, in the sense that some of its columns may be zero – actually ψ itself can turn out to be the zero matrix for that matter!

On the bright side however, one has:

Lemma 2.13. *If the weak Jacobian matrix associated to a set of bihomogeneous minimal generators of $(\mathcal{J}_{1,*})$ has rank over S then the weak Jacobian matrix associated to any other set of bihomogeneous minimal generators of $(\mathcal{J}_{1,*})$ has rank over S and the two ranks coincide.*

Proof. One shows, namely, that two such matrices have the same column submodule over S .

Let $\{P_1, \dots, P_s\}, \{Q_1, \dots, Q_s\} \subset k[\mathbf{X}, \mathbf{Y}]$ denote liftings of two sets of minimal bifurms of various bidegrees $(1, q)$, for suitable $q \geq 1$, generating $(\mathcal{J}_{1,*})$. Note that though the sets may be different, the set of corresponding bidegrees is uniquely defined up to reordering of generators. Say, $\deg(P_t) = \deg(Q_t) = (1, d_t)$, with $1 \leq t \leq s$ and $1 \leq d_1 \leq d_2 \leq \dots \leq d_s$. Therefore, for $t = 1, \dots, s$ we may write

$$Q_t = \sum_{j=1}^s \lambda_{tj} p_{tj}(\mathbf{Y}) P_j + \sum_{|\alpha_t|=d_t} a_{\alpha_t}(\mathbf{X}) \mathbf{Y}^{\alpha_t},$$

for suitable $\lambda_{tj} \in k$, $p_{tj}(\mathbf{Y}) \in k[\mathbf{Y}]$ of degree $d_t - \deg_{\mathbf{Y}}(P_j)$ and $a_{\alpha_t}(\mathbf{X}) \in \mathfrak{a}_1$. Setting $a_{\alpha_t}(\mathbf{X}) = \sum_{k=1}^r \mu_{tk} \ell_k(\mathbf{X})$, we obtain

$$\frac{\partial Q_t}{\partial X_i} = \sum_{j=1}^s \lambda_{tj} p_{tj}(\mathbf{Y}) \frac{\partial P_j}{\partial X_i} + \sum_{|\alpha_t|=d_t} \left(\sum_{k=1}^r \mu_{tk} \mathbf{Y}^{\alpha_t} \frac{\partial \ell_k(\mathbf{X})}{\partial X_i} \right).$$

This shows that the t th column of the Jacobian matrix of $\{\ell_1, \dots, \ell_r, Q_1, \dots, Q_s\}$ is contained in the column $k[\mathbf{Y}]$ -submodule of the Jacobian matrix of $\{\ell_1, \dots, \ell_r, P_1, \dots, P_s\}$. By symmetry, the two column $k[\mathbf{Y}]$ -submodules coincide, hence the S -submodule $\text{Im}(\psi)$ through any one of the two sets of generators is the same, as was to be shown. $\square$

Because of the previous lemma, we say that any weak Jacobian matrix associated to a set of minimal generators of $(\mathcal{J}_{1,*})$ is a JACOBIAN DUAL MATRIX of $\mathbf{f}$ (or of the corresponding m -targeted rational map $\mathfrak{F}$). The lemma unveils a numerical character of such a construction.

Definition 2.14. If some (hence, every) Jacobian dual matrix of an m -targeted rational map $\mathfrak{F} : \mathbf{f} \subset R \simeq k[\mathbf{X}]/\mathfrak{a}$ has a rank over S , we will say that $\mathfrak{F}$ has a JACOBIAN DUAL RANK and write $\text{jd rank}(\mathfrak{F})$ to denote it. In this situation, the NON-DEGENERATE JACOBIAN DUAL RANK of $\mathfrak{F}$ is $\text{jd rank}_+(\mathfrak{F}) := \text{jd rank}(\mathfrak{F}) - \text{dgi}(\mathfrak{a}) \geq 0$.

Note that the non-degenerate Jacobian dual rank is well defined, provided the Jacobian dual rank is.

The next result shows that the rank just introduced is sensitive to the dimension difference between source and target.

Proposition 2.15. Let $\mathfrak{F} : \mathbf{f} \subset R$ stand for an m -targeted rational map. Set $S = k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}] \subset R$. If $\mathfrak{F}$ has a Jacobian dual rank then

$$\dim R - \dim S \leq n - \text{jd rank}(\mathfrak{F}) = \text{edim}(R) - 1 - \text{jd rank}_+(\mathfrak{F}).$$

Proof. Up to a non-singular k -linear map on R , one may assume that the forms in $\mathbf{f}$ involve effectively at most the first $r+1 = \text{edim}(R)$ elements $x_0, \dots, x_r$. More precisely, let

$$A = \left(\begin{array}{c|c} \mathbf{1}_{r+1} & \mathbf{a} = (a_{h,i}) \\ \hline & \mathbf{1}_{n-r} \end{array} \right)$$

be an invertible k -matrix such that $(X_0, \dots, X_n) \cdot A = (X_0, \dots, X_r, \ell_{r+1}, \dots, \ell_n)$, with $\ell_h = X_h - \sum_{i=0}^r a_{h,i} X_i$, $r+1 \leq h \leq n$ and $\{\ell_{r+1}, \dots, \ell_n\}$ forming a basis of the k -vector space $\mathfrak{a}_1$. This defines a homogeneous k -algebra isomorphism

$$R = k[X_0, \dots, X_n]/\mathfrak{a} \simeq \tilde{R} = k[X_0, \dots, X_r]/\tilde{\mathfrak{a}}$$

which in turn restricts to homogeneous k -algebra isomorphism of the respective subalgebras $k[\mathbf{f}]$ and $k[\tilde{\mathbf{f}}]$, where $f_j \mapsto \tilde{f}_j$ under this map – however, note that the last isomorphism is not induced by equivalence of representatives, one is rather changing the rational map by “source and target” transformations.

Thus, this operation does not affect either $\dim R$ or $\dim S$ and, likewise, will not affect the numbers in the rightmost side of the stated inequality.

Updating the notation, consider the S -module $E := \operatorname{coker}_S(\psi^t)$ defined as the cokernel of the dual map associated to ψ . Since ψ^t has a rank over S , so does E . Then

$$\dim \mathcal{R}_S(E) = \dim S + \operatorname{rank}_S(E) = \dim S + \operatorname{edim}(R) - \operatorname{rank}_S(\psi),$$

where $\mathcal{R}_S(E)$ stands for the Rees algebra of E in the sense of [13]. On the other hand, a piece of the assumption on the base ideal $(\mathbf{f})$ implies that $\dim \mathcal{R}_R((\mathbf{f})) = \dim R + 1$ – for which it suffices to know that $(\mathbf{f})$ is not contained in at least one minimal prime of R of maximal dimension.

Then the asserted inequality follows from the inequality

$$\dim \mathcal{R}_R((\mathbf{f})) \leq \dim \mathcal{R}_S(E),$$

which in turn stems from the existence of an S -algebra surjection $\mathcal{R}_S(E) \twoheadrightarrow \mathcal{R}_R((\mathbf{f}))$, with $\mathcal{R}_R((\mathbf{f}))$ viewed as an S -algebra via

$$S = k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{Y}]/(\mathcal{J}_{0,*}) \hookrightarrow R[\mathbf{Y}]/\mathcal{J},$$

where $\mathcal{J}$ is the presentation ideal of $(\mathbf{f})$ based on these generators.

To see this, one first shows that there is an S -surjection $\mathcal{S}_S(E) \twoheadrightarrow \mathcal{R}_R((\mathbf{f}))$ with source the symmetric algebra of E . Indeed, one has

$$\begin{aligned} \mathcal{S}_S(E) &\simeq S[\mathbf{X}]/I_1(\mathbf{X} \cdot \psi^t) = k[\mathbf{Y}][\mathbf{X}]/(\mathfrak{b}, I_1(\mathbf{X} \cdot \psi^t)) \twoheadrightarrow R[\mathbf{Y}](\mathfrak{b}, I_1(\mathbf{X} \cdot \psi^t)) \\ &\twoheadrightarrow R[\mathbf{Y}]/\mathcal{J}. \end{aligned}$$

According to [13], $\mathcal{R}_S(E) \simeq \mathcal{S}_S(E)/(0 :_{\mathcal{S}_S(E)} s)$, the symmetric algebra modulo torsion, where s is a suitable regular homogeneous element of S . Therefore, it suffices to show that $0 :_{\mathcal{S}_S(E)} s$ maps to zero under the S -surjection $\mathcal{S}_S(E) \twoheadrightarrow \mathcal{R}_R((\mathbf{f}))$. Thus, pick an element $G \in 0 :_{\mathcal{S}_S(E)} s$ that maps to a nonzero element of $\mathcal{R}_R((\mathbf{f}))$. Then $s \in S \subset \mathcal{R}_R((\mathbf{f}))$ under the above injection annihilates a nonzero element of $\mathcal{R}_R((\mathbf{f}))$. Since $\mathcal{R}_R((\mathbf{f}))$ is R -torsionfree and S -torsionfree as a piece of the assumption that the inclusion $S \subset R$ is induced by a rational map, then $\mathcal{R}_R((\mathbf{f}))$ is S -torsionfree. Therefore, we get a contradiction. $\square$

Corollary 2.16. *With the notation of the previous theorem, $\operatorname{jd} \operatorname{rank}_+(\mathfrak{F}) \leq \operatorname{edim}(R) - 1$. Moreover, if equality holds then $\dim S = \dim R$.*

Thus, if the rank exists, one has the bounds:

$$\begin{aligned} \operatorname{dgi}(\mathfrak{a}) &\leq \operatorname{jd} \operatorname{rank}(\mathfrak{F}) \leq n, \\ 0 &\leq \operatorname{jd} \operatorname{rank}_+(\mathfrak{F}) \leq \operatorname{edim}(R) - 1. \end{aligned}$$

Remark 2.17. We note that, in the classical integral case in characteristic zero, the corollary says: $\operatorname{jd} \operatorname{rank}_+(\mathfrak{F}) = \operatorname{edim}(R) - 1$ implies that the general fiber of the map is a finite set of points. Thus, the preliminary estimate in Proposition 2.15 puts in evidence that the Jacobian dual rank plays a sort of complementary role to a transcendence degree in the integral case. The subsequent results will actually show that it takes over a complementary role to the degree of the rational map in the classical setup.

2.4. Statement and proof of the criterion

We now state the main theorem of this section, which gives a criterion of birationality. After having worked all this way a purely algebraic setup, it is natural to first convey an algebraic formulation of the criterion.

We keep the essential notation of the previous subsections.

Theorem 2.18. *Let $\mathfrak{F} : \mathbf{f} \subset R = k[\mathbf{X}]/\mathfrak{a} = k[X_0, \dots, X_n]/\mathfrak{a}$ stand for an m -targeted rational map and, for every minimal prime P of R , let $\mathfrak{F} \upharpoonright P$ denote the induced m -targeted rational map with source R/P . The following conditions are equivalent:*

- (a) $\mathfrak{F}$ is birational onto the image.
- (b) $\mathfrak{F}$ has non-degenerate Jacobian dual rank satisfying

$$\text{jd rank}_+(\mathfrak{F}) = \text{edim}(R) - 1.$$

- (c) For every minimal prime P of R , $\mathfrak{F} \upharpoonright P$ has non-degenerate Jacobian dual rank satisfying

$$\text{jd rank}_+(\mathfrak{F} \upharpoonright P) = \text{edim}(R/P) - 1.$$

SUPPLEMENT. *Suppose, moreover, that the above equivalent conditions holds. Let ψ denote a weak Jacobian dual matrix of $\mathfrak{F}$ over $S = k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}]$.*

- (i) *A rational $(n+1)$ -datum in S representing the inverse map is given by the coordinates of any homogeneous vector of positive degree in the (rank one) null space of ψ over S for which these coordinates generate an ideal containing a regular element.*
- (ii) *If further R is a domain, a rational $(n+1)$ -datum as in (i) can be taken to be the set of the (ordered, signed) $(\text{edim}(R) - 1)$ -minors of an arbitrary $(\text{edim}(R) - 1) \times \text{edim}(R)$ submatrix of ψ of rank $\text{edim}(R) - 1$.*

Proof. As in the proof of Proposition 2.15, up to a non-singular k -linear map on R , one may assume that the forms in $\mathbf{f}$ involve only the first $r+1 = \text{edim}(R)$ elements $x_0, \dots, x_r$. In addition, the resulting isomorphisms at the respective levels of R and S induce homogeneous k -algebra isomorphisms $K(R) \simeq K(\tilde{R})$ and $K(S) \simeq K(\tilde{S})$, where as before $S = k[\mathbf{Y}]/\mathfrak{b} \simeq k[\mathbf{f}] \simeq k[\tilde{\mathbf{f}}]$.

Finally, passing to degree zero and using the field criterion of Proposition 2.11, it follows that $\mathfrak{F} : \mathbf{f} \subset R$ is birational with source $\text{Proj}(R)$ if and only if $\tilde{\mathbf{f}} \subset \tilde{R}$ gives a birational map with source $\text{Proj}(\tilde{R})$. This shows that (a) is stable under this change. According to Proposition 2.15 and the discussion of its preliminaries, the non-degenerate Jacobian dual rank in both (a) and (b) is also stable under this process.

Thus, we may restart now assuming that $n+1 = \text{edim}(R)$ – in other words, $R = k[X_0, \dots, X_n]/\mathfrak{a}$, with $\mathfrak{a} \subset (R_+)^2$ – and $\text{jd rank}(\mathfrak{F}) = \text{jd rank}_+(\mathfrak{F})$.

(a) $\Rightarrow$ (b) By assumption, there is a rational $(n+1)$ -datum $\mathbf{g}(\mathbf{y}) = \{g_0(\mathbf{y}), \dots, g_n(\mathbf{y})\} \subset S$ such that $\mathbf{g}(\mathbf{f})$ is equivalent to the identity on R . Now, there is an isomorphism of k -algebras $\Phi : \mathcal{R}_R((\mathbf{f})) \simeq \mathcal{R}_S((\mathbf{g}))$ induced by the identity map of $k[\mathbf{X}, \mathbf{Y}]/(\mathfrak{a}, \mathfrak{b})$. The existence of this isomorphism is the same as in [9, Theorem 2.1, (i) $\Rightarrow$ (ii)], observing that the hypothesis of R (hence S) being a domain is superfluous because the two maps constructed in [9] were inverse

of each other by construction. We repeat the basic idea for the sake of completeness and further clarity while adapting the argument to the present context. One has isomorphisms

$$\frac{k[\mathbf{x}, \mathbf{y}]/(\mathfrak{a}, \mathfrak{b})}{\mathcal{J}_{\mathbf{f}}} \simeq \frac{k[\mathbf{x}]}{\mathfrak{a}}[\mathbf{f}t] = \mathcal{R}_R((\mathbf{f})), \quad \text{mapping } x_i \mapsto x_i \pmod{\mathfrak{a}}, y_j \mapsto f_j t$$

and, similarly,

$$\frac{k[\mathbf{x}, \mathbf{y}]/(\mathfrak{a}, \mathfrak{b})}{\mathcal{J}_{\mathbf{g}}} \simeq \frac{k[\mathbf{y}]}{\mathfrak{b}}[\mathbf{g}u] = \mathcal{R}_S((\mathbf{g})), \quad \text{mapping } y_i \mapsto y_i \pmod{\mathfrak{b}}, x_i \mapsto g_i u.$$

Then Φ drives $\mathcal{J}_{\mathbf{f}}$ inside $\mathcal{J}_{\mathbf{g}}$ by showing that if $F(\mathbf{x}, \mathbf{y}) \in k[\mathbf{x}, \mathbf{y}]$ is bihomogeneous such that $F(\mathbf{x}, \mathbf{f}t) \equiv 0 \pmod{\mathfrak{a}} k[\mathbf{x}, t]$ then $F(\mathbf{g}u, \mathbf{y}) \equiv 0 \pmod{\mathfrak{b}} k[\mathbf{y}, u]$. To prove this one uses that the composite $\mathbf{f}(\mathbf{g})$ is equivalent to the identity of $\text{Proj}(S)$. The reverse map works the same way by the obvious exchange.

Restricting to the bihomogeneous pieces, one has in particular $\Phi((\mathcal{J}_{\mathbf{f}})_{(1,s)}) = (\mathcal{J}_{\mathbf{g}})_{(s,1)}$, for every $s \geq 1$. Note that for varying $s \geq 1$, $(\mathcal{J}_{\mathbf{g}})_{(s,1)}$ generate the presentation ideal of the symmetric algebra of $\mathbf{g}$ over S . Up to a permutation of the variables $\mathbf{x}$, one has, say, $(\mathcal{J}_{\mathbf{g}})_{(s,1)} = I_1((x_0 \cdots x_{\mu-1}) \cdot \mathcal{M})$, where $\mathcal{M}$ denotes the matrix of a minimal graded presentation of $\mathbf{g}$ over S and $\mu = \mu(\mathbf{g})$ is the minimal number of generators of the ideal $(\mathbf{g})$. Further, if ψ denotes a weak Jacobian dual matrix of $\mathbf{f}$ over S then $\Phi((\mathcal{J}_{\mathbf{f}})_{(1,s)}) = I_1((x_0 \cdots x_n) \cdot \psi^t)$ by the definition of ψ . The ensued equality $I_1((x_0 \cdots x_n) \cdot \psi^t) = I_1((x_0 \cdots x_{\mu-1}) \cdot \mathcal{M})$ shows that the last $n - \mu$ rows of ψ^t are null. Hence if ψ' is the matrix obtained by removing the zero columns of ψ then any column of ψ'^t lives in $\text{Im}(\mathcal{M})$ and vice versa. Therefore $\text{Im}(\psi') = \text{Im}(\mathcal{M})$.

Now $\mathcal{M}$ has a rank over S . This is because $\text{grade}_S(\mathbf{g}) \geq 1$ by definition of a rational datum, hence a presentation map of $(\mathbf{g})$ over S has a rank as well. Clearly then ψ' has a rank and $\text{rank}(\psi) = \text{rank}(\psi') = \text{rank}(\mathcal{M}) = \mu - 1 = \text{edim}(R) - 1$, the last equality following from the non-degeneracy of R .

(b) $\Rightarrow$ (c) Note that $n = \text{edim}(R) - 1$ as we are assuming that R is non-degenerate. By the definition of rank, the ideal of minors $I_n(\psi)$ of a weak Jacobian dual matrix ψ has a regular element on S . Let P be minimal prime of R and $Q = P \cap S$, using the isomorphism $S \simeq k[\mathbf{f}]$. Clearly then $I_n(\psi \upharpoonright Q) \neq \{0\}$, where $\psi \upharpoonright Q$ is the matrix on S/Q obtained by taking the entries of ψ modulo Q . It follows that $\psi \upharpoonright Q$ has rank n over S/Q .

Now consider a weak Jacobian dual matrix $\tilde{\psi}$ of $\mathfrak{F} \upharpoonright P$, as obtained from the Rees algebra $\mathcal{R}_{R/P}((\mathbf{f}) + P/P)$. By definition, $\text{Im}(\psi \upharpoonright Q)$ can be taken to be a submodule of $\text{Im}(\tilde{\psi})$. Therefore, the rank of $\tilde{\psi}$ is at least n , hence it obtains

$$\text{edim}(R/P) - 1 - \text{jdrank}_+(\mathfrak{F} \upharpoonright P) = n - \text{jdrank}(\tilde{\mathfrak{F}} \upharpoonright P) \leq 0,$$

from which ensues $\text{jdrank}_+(\mathfrak{F} \upharpoonright P) \geq \text{edim}(R/P) - 1$ and then equality follows by Proposition 2.15, as required.

(c) $\Rightarrow$ (a) The proof will simultaneously argue for item (ii).

The first step is to use Proposition 2.11, by which the condition in (a) is equivalent to the condition that every restriction $\mathfrak{F} \upharpoonright P$ of $\mathfrak{F}$ is birational onto its image and, moreover, R/P is the unique minimal component of R mapping to the latter by $\mathfrak{F}$.

To show that one can reduce to the case where R is a domain, we argue as follows. Given a minimal prime P of R , write $R/P \simeq (k[\mathbf{x}]/\mathfrak{a})/(\mathfrak{P}/\mathfrak{a}) \simeq k[\mathbf{x}]/\mathfrak{P}$ and $S/P \cap S \simeq k[\mathbf{y}]/\mathfrak{Q}$, where $\mathfrak{Q}$ is the inverse image of $\mathfrak{P}$ in $k[\mathbf{y}]$. Further, set $\mathfrak{F}_{\mathfrak{P}} = \mathfrak{F}|_P$ to emphasize the original polynomial ambient. The assumption then reads as

$$\mathrm{jd\,rank}_+(\mathfrak{F}_{\mathfrak{P}}) = \mathrm{edim}(R/P) - 1 = n - \mathrm{dgi}(\mathfrak{P}),$$

where $\mathrm{dgi}(\mathfrak{P})$ is the degeneration index of $\mathfrak{P}$ as discussed in the beginning of the subsection. Equivalently, $\mathrm{jd\,rank}(\mathfrak{F}_{\mathfrak{P}}) = n$. Thus, if we show that this implies that $\mathfrak{F}_{\mathfrak{P}}$ is birational onto its image we will be done.

Therefore, we may assume at the outset that R is a domain and a weak Jacobian dual matrix ψ has rank n . But then one can select an $n \times (n+1)$ submatrix σ of ψ of rank n . Let $\Delta := \{\Delta_0, \dots, \Delta_n\}$ stand for the ordered signed n -minors of σ . This set is a rational $(n+1)$ -datum. It remains to prove that $\Delta(\mathbf{f})$ is equivalent to the identity on $\mathrm{Proj}(R)$.

Apply Proposition 2.1 with $A := k[\mathbf{y}]$ and with ρ standing for the transpose of the selected $n \times (n+1)$ submatrix σ of ψ of rank n to get that the Koszul forms $\Delta_i x_j - \Delta_j x_i$, one for each pair i, j , belong to the $k[\mathbf{y}]$ -submodule of $A^{n+1} = \sum_{i=0}^n A x_i$ generated by the forms induced by the rows of σ – i.e., obtained as inner product of these by the transpose $(x_0, \dots, x_n)^t$. But the latter, by definition, belong to the defining ideal of the Rees algebra of $(\mathbf{f})$ over $R[\mathbf{Y}]$, hence vanish on the tuple whose coordinates are the constituents of $\mathbf{f}$.

Therefore

$$\Delta_i(\mathbf{f})x_j = \Delta_j(\mathbf{f})x_i, \quad \text{for each pair } i, j. \quad (3)$$

Recall that some $\Delta_i(\mathbf{f}) \neq 0$; say, this is the case for $i = n$. On the other hand, $x_n \neq 0$ since otherwise (3) implies that $x_i = 0$ for every i , which is nonsense. In particular, one has the equalities $\Delta_n(\mathbf{f})x_i = x_n \Delta_i(\mathbf{f})$, with $\Delta_n(\mathbf{f}), x_n$ nonzero. Therefore $\Delta(\mathbf{f})$ is equivalent to the identity on $\mathrm{Proj}(R)$.

It remains to prove (i).

Let $\mathbf{h}^t \in \ker(\psi^t)$ be given as in the hypothesis of (i). Let $\mathbf{g}$ be a representative of the inverse map such that the vector $\mathbf{g}^t \in \ker(\psi^t)$ as in the proof and notation of the implication (a) $\Rightarrow$ (b). The argument there showed that $\ker(\psi^t)$ has rank 1. Since the ideal $(\mathbf{h})$ has a regular element by assumption, Lemma 2.5 implies that $\mathbf{h}$ and $\mathbf{g}$ are equivalent $(n+1)$ -sets. Then, by Lemma 2.6, $\mathbf{h}$ is rational datum, hence defines the inverse map as well. $\square$

Remark 2.19. The argument in the proof of (ii) above goes through regardless of the integrality assumption, provided a weak Jacobian dual matrix of $\mathbf{f}$ admits a submatrix of size $(\mathrm{edim}(R) - 1) \times \mathrm{edim}(R)$ and of rank $\mathrm{edim}(R) - 1$.

We now transcribe the above result into a geometric version. We assume that the ground field k is infinite and that varieties have points in an algebraic closure of k .

Theorem 2.20. Let $V \subset \mathbb{P}^n$ ($n \geq 1$) denote a reduced, but not necessarily irreducible, projective subvariety. Let $\mathfrak{F} : V \dashrightarrow \mathbb{P}_k^m$ denote a rational map.

The following conditions are equivalent:

- (a) $\mathfrak{F}$ is a birational map of V onto its image.
- (b) $\mathfrak{F}$ has non-degenerate Jacobian dual rank and the latter equals the embedding dimension of the affine cone over V .
- (c) For every irreducible component W of V , the restriction of $\mathfrak{F}$ to W has non-degenerate Jacobian dual rank and the latter equals the embedding dimension of the affine cone over W .

Moreover, when these conditions hold then

- (i) The inverse map to $\mathfrak{F}$ is given by the coordinates of any sufficiently general homogeneous vector of positive degree in the null space of a weak Jacobian dual matrix of $\mathfrak{F}$.
- (ii) If besides, V is irreducible, the coordinates of the inverse to $\mathfrak{F}$ are the (ordered, signed) n -minors of an arbitrary $n \times (n+1)$ submatrix rank n of a weak Jacobian dual matrix of $\mathfrak{F}$.

(“Sufficiently general” in the above statement means that the general k -linear combination of the coordinates of the vector ought to be a nonzero-divisor on the homogeneous coordinate ring of the image.)

Remark 2.21. (1) A tall order on the computational side of the theory is to get hold of the spanning biforms of $\mathcal{J}_{1,*}$ and $\mathcal{J}_{0,*}$ without actually computing all the generators of $\mathcal{J}_{\mathbf{f}}$. For the second bihomogeneous piece this is beckoningly the main problem of implicitization in elimination theory. One of the common techniques is to try and derive the implicit equations by means of the so-called Sylvester method starting out from the defining equations of the symmetric algebra of $I = (\mathbf{f})$ (i.e., the bihomogeneous piece of bidegrees $(*, 1)$). In special situations, this method often throws some light on obtaining $\mathcal{J}_{1,*}$ – see, e.g., [5, Example 5.7].

(2) Note that the theorem gives no hint on the minimal possible degree of a representative of the inverse map. Even in the case of a Cremona map, the degree will be elusive as the representative given by the minors will typically carry a non-trivial factor. Thus, the question arises as to how one goes about computing the degree of the gcd of the minors in the latter case. Often this can be computed by a careful scrutiny of the Hilbert series (see, e.g., [8, Proposition 3.1]) or by local structure (see [8, Examples 2.4, 2.5,] and also [17, Theorem 3.1]).

(3) It would be curious to understand the meaning of a weak Jacobian dual matrix – or at least of its rank – in purely geometric terms. In particular, the Jacobian dual rank seems to be related to the field degree in a sort of complementarity.

3. Tidbits on birational maps

3.1. Algebraic preliminaries

We recall a few algebraic notions which will be used in this part.

A very basic condition on an ideal is as follows.

Definition 3.1. An ideal $I \subset R$ in a ring R is said to be OF LINEAR TYPE if the natural A -algebra homomorphism $S_R(I) \rightarrow \mathcal{R}_R(I)$ is injective.

Here $\mathcal{S}_R(I)$ and $\mathcal{R}_R(I)$ denote the symmetric and the Rees algebra of I , respectively. The literature on this is quite extensive (see [10,11,13] and the book [18]).

Now assume that $R = k[\mathbf{x}] = k[X_0, \dots, X_n]/\mathfrak{a}$ denotes a standard graded k -algebra over a field k .

Let $\mathbf{f} = \{f_0, \dots, f_m\} \subset R$ be forms of a fixed degree $d \geq 1$. We will henceforth denote by $\varphi = \varphi_{\mathbf{f}}$ a graded matrix whose columns generate the first syzygy module of $\mathbf{f}$. Thus, one has a free graded presentation of the homogeneous ideal $I := (\mathbf{f}) \subset R$,

$$\bigoplus_s R(-d_s) \xrightarrow{\varphi} R^m(-d) \rightarrow R. \quad (4)$$

We denote by $\varphi_1 = \varphi_1(\mathbf{f})$ the matrix whose columns form a set of generators of the submodule $\text{Im}(\varphi)_{d+1}$ (i.e., the coordinates of such vectors as elements of the standard graded ring R are linear forms). Clearly, φ_1 can always be taken as a submatrix of φ , so we informally say that φ_1 is the LINEAR PART of the whole syzygy matrix φ . Not infrequent is the case where $\varphi_1 = 0$. On the other end of the spectrum, an important special case is when $\varphi_1 = \varphi$ - we then say that I has a LINEAR PRESENTATION.

A weaker condition is a certain rank requirement. Recall that φ has a well-defined rank r if and only if $I_s(\varphi) = \{0\}$ for $s > r$ and $I_r(\varphi)$ has a regular element. This is equivalent to requiring that the R -submodule $Z := \text{Im}(\varphi) \subset R^m$ have rank r - i.e., that Z_p be free of rank r for every associated prime of R . One also has that Z (hence, φ too) has a rank if and only if I has a regular element.

We will say that φ_1 has maximal (possible) rank if φ has a rank and $\text{rank}(\varphi_1) = \text{rank}(\varphi)$. We could also say in this situation that φ has MAXIMAL LINEAR RANK.

3.2. Consequences of the main criterion

In this part we draw some corollaries of Theorem 2.18 in the case where $\mathfrak{F}: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Recall that a birational map of $\mathbb{P}^n$ onto itself is known as a CREMONA TRANSFORMATION.

We emphasize that in this context much of the theory, including the main criterion, gets simplified and more visible. Cremona transformations have a uniquely defined minimal representative (both the map and its inverse) and the corresponding degrees are important invariants in the classical theory.

Some of the applications in this section envisage solving the questions posed in [9], in that they are characteristic-free.

For example, one has the following characteristic-free version of [9, Theorem 4.1] and [2, Proposition 4.5(iii)].

Theorem 3.2. *Let $\mathfrak{F}: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map, given by $m+1$ forms $\mathbf{f} = \{f_0, \dots, f_m\}$ of a fixed degree. If the image of $\mathfrak{F}$ has dimension n and $\text{rank } \varphi_1 = n$ (maximal possible) then $\mathfrak{F}$ is birational onto its image.*

Proof. Following the basic *leitmotif* of [10, Proposition 1.1], write $\mathbf{Y} \cdot \varphi_1 = \mathbf{X} \cdot \rho$, where ρ is a uniquely defined matrix whose entries are linear forms in $k[\mathbf{Y}] = k[Y_0, \dots, Y_m]$. Note that the left (respectively, right) side generates the presentation ideal of the symmetric algebra $\mathcal{S}_{k[\mathbf{X}]}(\text{coker}(\varphi_1))$ (respectively, $\mathcal{S}_{k[\mathbf{Y}]}(\text{coker}(\rho))$). Therefore, the identity map of $k[\mathbf{X}, \mathbf{Y}]$ induces a k -isomorphism between these algebras.

Note the surjection $\text{coker}(\varphi_1) \twoheadrightarrow I = (\mathbf{f})$, whose kernel is torsion since φ_1 has rank n by hypothesis. Therefore

$$\mathcal{R}_{k[\mathbf{X}]}(\text{coker}(\varphi_1)) \simeq \mathcal{R}_{k[\mathbf{X}]}(I). \quad (5)$$

Let $\mathcal{J}$ denote the presentation ideal of the Rees algebra $\mathcal{R}_{k[\mathbf{X}]}(I)$ on $k[\mathbf{X}, \mathbf{Y}]$. Write $S := k[\mathbf{Y}]/\mathfrak{b}$ for the homogeneous coordinate ring of the image of $\mathfrak{F}$ on $\mathbb{P}^m$ and. Recall from previous sections that $\mathfrak{b} \subset \mathcal{J}$.

Write $E := \text{coker}(\rho)$. Since the symmetric algebra is functorial, one has

$$S_S(E \otimes_{k[\mathbf{Y}]} S) \simeq S_{k[\mathbf{Y}]}(E)/\mathfrak{b}S_{k[\mathbf{Y}]}(E) \simeq k[\mathbf{X}, \mathbf{Y}]/(\mathfrak{b}, I_1(\mathbf{X} \cdot \rho)) = k[\mathbf{X}, \mathbf{Y}]/(\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1)).$$

Thus, there is a surjection $S_S(E \otimes_{k[\mathbf{Y}]} S) \twoheadrightarrow k[\mathbf{X}, \mathbf{Y}]/\mathcal{J}$.

Claim. *The surjection $S_S(E \otimes_{k[\mathbf{Y}]} S) \twoheadrightarrow k[\mathbf{X}, \mathbf{Y}]/\mathcal{J}$ induces an isomorphism*

$$\mathcal{R}_S(E \otimes_{k[\mathbf{Y}]} S) \simeq k[\mathbf{X}, \mathbf{Y}]/\mathcal{J}.$$

Recalling that the Rees algebra $\mathcal{R}_S(E \otimes_{k[\mathbf{Y}]} S)$ is obtained thereof by killing the S -torsion, we write $\mathcal{T}$ for the lifting of the latter to $k[\mathbf{X}, \mathbf{Y}]$. Thus

$$\mathcal{R}_S(E \otimes_{k[\mathbf{Y}]} S) \simeq k[\mathbf{X}, \mathbf{Y}]/(\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1), \mathcal{T}).$$

Then let $G \in \mathcal{T}$. Then $F(\mathbf{Y})G \in (\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1)) \subset \mathcal{J}$ for some $F(\mathbf{Y}) \in k[\mathbf{Y}] \setminus \mathfrak{b}$. Since $\mathcal{J}$ is a prime ideal and $F(\mathbf{Y}) \notin \mathcal{J}$ because $k[\mathbf{Y}] \cap \mathcal{J} = \mathfrak{b}$, necessarily $G \in \mathcal{J}$. This proves the inclusion $(\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1), \mathcal{T}) \subset \mathcal{J}$ as $(\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1)) \subset \mathcal{J}$.

Conversely, let $G = G(\mathbf{X}, \mathbf{Y}) \in \mathcal{J}$. By (5), $\mathcal{J}$ is the $k[\mathbf{X}]$ -torsion of $k[\mathbf{X}, \mathbf{Y}]/I_1(\mathbf{Y} \cdot \varphi_1)$ lifted to $k[\mathbf{X}, \mathbf{Y}]$. Therefore, there exists $F(\mathbf{X}) \in k[\mathbf{X}] \setminus I_1(\mathbf{Y} \cdot \varphi_1)$ – i.e., $F(\mathbf{X}) \neq 0$ – such that $F(\mathbf{X})G(\mathbf{X}, \mathbf{Y}) \in I_1(\mathbf{Y} \cdot \varphi_1)$. Suppose that $G \notin (\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1), \mathcal{T})$. The latter is a prime ideal, hence $F(\mathbf{X}) \in (\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1), \mathcal{T})$. By the definition of $\mathcal{T}$ there exists $H(\mathbf{Y}) \in k[\mathbf{Y}] \setminus \mathfrak{b}$ such that $H(\mathbf{Y})F(\mathbf{X}) \in (\mathfrak{b}, I_1(\mathbf{Y} \cdot \varphi_1))$. Evaluating $\mathbf{Y} \mapsto \mathbf{f}$ would give $H(\mathbf{f})F(\mathbf{X}) = 0$ whence $F(\mathbf{X}) = 0$ since $H(\mathbf{f}) \neq 0$; this is a contradiction.

To conclude we count dimensions according to [13], thereby finding

$$\dim S + \text{rank}_S(E \otimes_{k[\mathbf{Y}]} S) = n + 1 + 1.$$

Since $\dim S = n + 1$ by assumption, we get $\text{rank}_S(E \otimes_{k[\mathbf{Y}]} S) = 1$, hence $\text{rank}_S(\rho \otimes_{k[\mathbf{Y}]} S) = n$. Since $\text{Im}(\rho \otimes_{k[\mathbf{Y}]} S) \subset \text{Im}(\psi)$ for a weak Jacobian dual matrix ψ over S , Theorem 2.18 applies. $\square$

Easy examples show that even if $m = n$ and the forms $\mathbf{f}$ generate a linearly presented ideal – i.e., φ is linear – they can turn out to be algebraically dependent. Perhaps the simplest is the map $\mathfrak{F} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by the 2-forms $X_0X_2, X_0X_3, X_1X_2, X_1X_3$. At the other end of the spectrum, it is obvious (and easy) that algebraic independence does not imply anything in particular concerning the rank of φ_1 . Therefore the two conditions of the proposition are pretty much mutually independent.

The following example gives an illustration of the above proposition.

Example 3.3. Consider the map $\mathfrak{F} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ defined by the 3-forms

$$X_0X_1X_2, X_0X_2X_3, X_0X_4X_5, X_1X_2X_4, X_2X_3X_5, X_3X_4X_5.$$

Since these are square-free monomials, it is fairly easy to compute the matrix φ_1 , which is here of maximal rank ($= 5$). In fact, φ_1 generates a free submodule. A computation with *Macaulay* [1] yields that (notation as above) the three algebras $\mathcal{S}(\text{coker}(\varphi_1))$, $\mathcal{S}(I)$ and $\mathcal{R}(I)$ are all different though of the same dimension. In particular, I is not of linear type. The base ideal of the inverse of $\mathfrak{F}$ is generated by 4-forms, so this degree is one less than the expected degree were I of linear type (cf. Theorem 3.4 and its supplement).

In the case of rational maps defined by monomials as this, there is a blind (i.e., algorithmic) way to both testing birationality and obtaining the coordinates of the inverse map (see [17]).

In the remaining of the section, following the classical terminology, we say that a Cremona map has degree r if it is defined by forms of degree r generating an ideal of codimension ≥ 2 .

One can prove the converse to the linear obstruction result ([9, Proposition 3.5]) in any characteristic.

Proposition 3.4. *Let $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map whose base ideal I is of linear type. The following conditions are equivalent:*

- (a) $\mathfrak{F}$ is a Cremona transformation.
- (b) The linear syzygy matrix φ_1 has rank n .

Moreover, if these conditions take place, the inverse map $\mathfrak{F}^{-1}$ has degree $\leq n$.

Proof. The implication (a) $\Rightarrow$ (b) has already been observed in [9, Proposition 3.5].

The reverse implication (b) $\Rightarrow$ (a) comes from Theorem 3.2 since the linear type assumption for an ideal generated by forms of the same degree implies at least the algebraic independence of these forms.

The last assertion follows from part (ii) of the supplement to Theorem 2.18. Indeed, the transpose of an $n \times (n+1)$ matrix of rank n formed by linear syzygies of I can be taken as a submatrix of a weak Jacobian dual matrix of $\mathfrak{F}$. Since the signed n -minors of this matrix define the inverse to $\mathfrak{F}$, the degree bound is clear. $\square$

The conjectured statements [9, Corollaries 3.8 through 3.10] become true statements in any characteristic which we briefly restate for convenience and reference.

Corollary 3.5. *Let $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map whose base ideal I is a one-dimensional locally complete intersection at its minimal primes. Then $\mathfrak{F}$ defines a Cremona transformation of $\mathbb{P}^n$ if and only if the linear syzygy matrix φ_1 has rank n .*

Proof. The result follows because the assumption in the one-dimensional case means that the ideal is of linear type (see [14, Proposition 3.7], also [4, Remark 10.5]). $\square$

In particular, this applies to the case of plane rational maps, thereby yielding a strong outcome:

Corollary 3.6. *Let $\mathfrak{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map whose base ideal scheme is a codimension 2 locally complete intersection. Then $\mathfrak{F}$ is a Cremona map if and only if is defined by the 2×2 minors of a 3×2 matrix of linear forms.*

Proof. Suppose that $\mathfrak{F}$ is Cremona. By Proposition 3.4, the inverse map is of degree ≤ 2 . Since we are in the plane case, $\mathfrak{F}$ has also degree ≤ 2 . Therefore, as the base ideal is of codimension 2, the map is one of the three classical types of quadratic Cremona maps, in which case the result is directly verified.

The converse follows directly from the previous corollary. $\square$

Question 3.7. *Let $f \in k[X_0, X_1, X_2]$ be a reduced homogeneous polynomial whose polar map is a Cremona map. Is the ideal generated by the partial derivatives of f locally a complete intersection at its minimal primes?*

The answer is positive in characteristic zero due to the theorem of Dolgachev [3]. However, even in null characteristic an independent proof would be of interest as it would imply Dolgachev's result by way of Corollary 3.6.

Remark 3.8. In the plane case ($n = 2$), the above local condition on the base ideal is also equivalent to requiring that its n th Fitting ideal $I_1(\varphi)$ have codimension three, so the condition is easily verified by computation.

3.3. The semilinear generation degree

In this part we raise a few questions related to certain special behavior of the rational map vis-à-vis its base ideal $I \subset R$.

Proposition 3.9. *Let $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map and let $W \subset \mathbb{P}^m$ denote its image. If the codimension of the ideal $(\mathcal{J}_{1,*})$ over the ring $(k[\mathbf{Y}]/I(W))[\mathbf{X}]$ is at least n then $\mathfrak{F}$ is birational onto W .*

Proof. By Theorem 2.20, it suffices to show that the rank of the Jacobian dual matrix ψ over $S := k[\mathbf{Y}]/I(W)$ is at least n (hence $= n$).

For this note that if $C := \text{coker}_S(\psi^t)$, then the symmetric algebra $\mathcal{S}_S(C) \simeq S[\mathbf{X}]/I_1(\mathbf{X} \cdot \psi^t)$. In addition, by the very definition, $(\mathcal{J}_{1,*})S[\mathbf{X}] = I_1(\mathbf{X} \cdot \psi^t)$. It follows that

$$\dim \mathcal{S}_S(C) \leq \dim S + n + 1 - \text{ht}((\mathcal{J}_{1,*})S[\mathbf{X}]) \leq \dim S + n + 1 - n = \dim S + 1.$$

On the other hand, $\dim \mathcal{S}_S(C) \geq \dim \mathcal{R}_S(C) = \dim S + \text{rank}_S(C)$, where $\mathcal{R}_S(C)$ stands for the Rees algebra of C . From this $\text{rank}_S(C) \leq 1$, hence $\text{rank}_S(\psi) = \text{rank}_S(\psi^t) \geq n$. $\square$

Regarding the above proof, one sees that since $\text{rank}_S(\psi) = n$ then actually $\text{rank}_S(C) = 1$. In this notation, the inverse map to $\mathfrak{F}$ is defined by the coordinates of any nonzero homogeneous vector in the dual module C^* .

The converse statement to the effect that $\text{rank}_S(\psi) = n$ over $k[\mathbf{Y}]/I(W)$ implies that the codimension of $(\mathcal{J}_{1,*})$ over $(k[\mathbf{Y}]/I(W))[\mathbf{X}]$ is at least n is generally false, even for Cremona maps. The reason is simply given by the fact that there are Cremona maps for which the symmetric algebra of the base ideal has larger dimension than its Rees algebra. No such examples exist for plane Cremona maps as the base ideal is an almost complete intersection. For space Cremona maps, however, even monomial involutions may show this behavior as the following example conveys.

Example 3.10. Let $\mathfrak{F} = (X^4 : X^2YW : XY^2Z : Y^3Z) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Then $(\mathcal{J}_{1,*}) \subset k[\mathbf{X}, \mathbf{Y}]$ has codimension 2, i.e., $\dim S(I) = 6$, while $\text{rank } \psi = 3$ as the map is Cremona. Here of course $\text{rank}(\varphi_1) < 3$ (actually $\text{rank}(\varphi_1) = 1$ by calculation). We note that this Cremona map is an involution up to permutation in the source and in the target. A different approach to verify birationality can be applied in this example since the base ideal is generated by monomials (see [17]).

In connection to Proposition 3.9, the following numerical invariant seems worth looking at.

Definition 3.11. The SEMILINEAR GENERATION DEGREE of an ideal $I \subset R = k[\mathbf{X}]$, denoted $\text{sgd}(I)$, is the minimum $s \geq 1$ such that $(\mathcal{J}_{1,*}) = (\mathcal{J}_{1,\leq s})$, where the ideal on the right side is the subideal generated by all (“semilinear”) relations of bidegree $(1, r)$ with $r \leq s$.

A souped-up version would require only $\text{ht}(\mathcal{J}_{1,*}) = \text{ht}(\mathcal{J}_{1,\leq s})$, in which case the invariant could be dubbed the SEMILINEAR CODIMENSION DEGREE.

This numerical invariant expresses the measure of how far off through the successive powers I^m of the ideal I one has to go in order to pick a sufficient number of linear syzygies of I^m to generate the ideal $(\mathcal{J}_{1,*})$. In case I is actually the base ideal of a Cremona map then this number may have bearing to the degree of the coordinates of the inverse map. Here is a very crude lower bound:

Proposition 3.12. Let $I \subset k[\mathbf{X}]$ denote the base ideal of a Cremona map $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then $\text{sgd}(I) \geq \deg(\mathfrak{F}^{-1})/n$, where $\deg \mathfrak{F}^{-1}$ denotes the degree of the defining coordinates of the inverse map.

Proof. Using item (ii) of Theorem 2.20 one immediately has $\deg \mathfrak{F}^{-1} \leq \text{sgd}(I)n$, as required. $\square$

For instance, if I is of linear type then $\deg \mathfrak{F}^{-1} \leq n$, a property already observed in Proposition 3.4.

Question 3.13. Find upper bounds for $\text{sgd } I$ in terms of other invariants of I .

A related question was raised by Vasconcelos in connection with elimination theory. Note that the above numerical invariant makes sense even for a general rational map with target some $\mathbb{P}^m$.

Question 3.14. Let more generally $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, given by $m+1$ k -linearly independent forms $\mathbf{f} = \{f_0, \dots, f_m\} \subset k[\mathbf{X}]$ of a fixed degree ≥ 2 , without proper common factor. Let $I = (f_0, \dots, f_m)$ denote as above the base ideal of $\mathfrak{F}$ and let d denote the highest degree of a minimal form defining the image of $\mathfrak{F}$ in $\mathbb{P}^m$. Is $\text{sgd}(I) \leq d$?

Note that the answer is trivially affirmative for ideals of fiber type. There is both theoretical and computational evidence for an affirmative answer in the case of an $\mathbf{X}$ -primary almost complete intersection (see [5]).

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