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# The Ribaucour Transformation for Hypersurfaces of Two Space Forms and Conformally Flat Hypersurfaces

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**Abstract** We develop a Ribaucour transformation for the class of conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures of a pseudo-Riemannian space form of dimension 4, constant curvature $c$ and index $s \in \{0, 1\}$, as well as for the class of hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures for which there exists another isometric immersion $\tilde{f}\colon M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $\tilde{c} \neq c$. It gives a process to produce a family of new elements of those classes starting from a given one and a solution of a linear system of PDE's. This enables us to construct explicit new examples of hypersurfaces in both classes.

**Keywords** Conformally flet hypersurfaces · Hypersurfaces of two space forms · Ribaucour transformation

## 1 Introduction

The study of conformally flat hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$ of dimension $n$ of Euclidean space is a classical topic in differential geometry initiated by Cartan (1917), who proved that they must have a principal curvature of multiplicity at least $n - 1$

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if $n \geq 4$. In other words, conformally flat Euclidean hypersurfaces of dimension $n \geq 4$ are generically envelopes of one-parameter families of hyperspheres. Cartan also observed that this is no longer true for $n = 3$, that is, that there exist conformally flat hypersurfaces $f: M^3 \to \mathbb{R}^4$ with three distinct principal curvatures.

The work by Cartan was taken up by Hertrich-Jeromin (1994), who showed that any conformally flat hypersurface $f: M^3 \to \mathbb{R}^4$ with three distinct principal curvatures admits locally principal coordinates $(u_1, u_2, u_3)$ such that the induced metric $ds^2 = \sum_{i=1}^{3} v_i^2 du_i^2$ satisfies the Guichard condition, say,

$$v_2^2 = v_1^2 + v_3^2.$$

Then he used the conformal invariance of this condition to associate to each such hypersurface a Guichard net in $\mathbb{R}^3$, that is, a conformally flat metric on an open subset of $\mathbb{R}^3$ satisfying the Guichard condition, which is unique up to a Möbius transformation. He also proved in Hertrich-Jeromin (1994) that the converse holds, that is, that each conformally flat 3-metric satisfying the Guichard condition gives rise to a unique (up to a Möbius transformation) conformally flat hypersurface in $\mathbb{R}^4$. In this way, the classifications of conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures and of conformally flat 3-metrics satisfying the Guichard condition are equivalent problems.

This point of view was pursued in some subsequent papers; see, for instance, Hertrich-Jeromin and Suyama (2007) (respectively, Hertrich-Jeromin and Suyama 2013), where a classification was given of conformally flat Euclidean hypersurfaces associated to cyclic (respectively, Bianchi-type) Guichard nets in $\mathbb{R}^3$, that is, Guichard nets in $\mathbb{R}^3$ for which one of the coordinate line families consists of circular arcs (respectively, the coordinate surfaces have constant sectional curvature).

Some significant advances on the understanding of the space of conformally flat 3-metrics satisfying the Guichard condition have been recently obtained in Burstall et al. (2018). Namely, for a conformally flat 3-metric with the Guichard condition in the interior of the space, an evolution of orthogonal Riemannian 2-metrics with constant Gauss curvature $-1$ was determined; conversely, for a 2-metric belonging to a certain class of orthogonal analytic 2-metrics with constant Gauss curvature $-1$, a one-parameter family of conformally flat 3-metrics with the Guichard condition was determined as evolutions issuing from the 2-metric.

However, it is not in general an easy task to translate results on conformally flat 3-metrics satisfying the Guichard condition to corresponding ones on their associated conformally flat Euclidean hypersurfaces. In fact, due to the difficulties involved in this approach, even the construction of further examples of conformally flat Euclidean hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures became a challenging problem. Recent progress in this direction was achieved in Hertrich-Jeromin et al. (2015) (see also Hertrich-Jeromin and Suyama 2015) by the discovery that each conformally flat Euclidean hypersurface has a dual one, which is related to it by a Combescure transformation, and this duality can be used to obtain new conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures from a given one.

It was recently shown in Canevari and Tojeiro (2017) that the existence of principal coordinates satisfying some additional conditions actually characterizes conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures of any pseudo-Riemannian space form $\mathbb{Q}_s^4(c)$ of dimension 4, constant sectional curvature $c$ and index $s \in \{0, 1\}$, that is, $\mathbb{Q}_s^4(c)$ is either a Riemannian or Lorentzian space-form of constant curvature $c$, corresponding to $s = 0$ or $s = 1$, respectively (see Theorem 2 below).

It was also shown in Canevari and Tojeiro (2017) that the class of conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures is closely related to the class of hypersurfaces of $\mathbb{Q}_s^4(c)$ that are solutions of the following natural problem:

*Problem* $*$: *For which hypersurfaces* $f\colon M^3 \to \mathbb{Q}_s^4(c)$ *does there exist another isometric immersion* $\tilde{f}\colon M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ *with* $\tilde{c} \neq c$?

In fact, a similar characterization was given of hypersurfaces of $\mathbb{Q}_s^4(c)$ that are solutions of Problem $*$ (see Theorem 3 below).

The aim of this paper is to use such characterizations to develop a Ribaucour transformation (see Sect. 3) for both classes of hypersurfaces. It yields a process to generate a family of new elements of such classes starting from a given one and a solution of a linear system of partial differential equations (see Theorem 9 below). In particular, explicit new examples of (one-parameter families of) conformally flat hypersurfaces of $\mathbb{R}^4$ with three distinct principal curvatures are constructed in Sect. 4, whose associated Guichard nets are neither cyclic nor of Bianchi-type, as can be easily checked by using the criteria in Hertrich-Jeromin and Suyama (2007) and Hertrich-Jeromin and Suyama (2013). We also produce explicit examples of (one-parameter families of) hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures that admit an isometric immersion into $\mathbb{Q}_s^4(\tilde{c})$ with $c \neq \tilde{c}$.

## 2 The Characterization of Conformally Flat Hypersurfaces and of Solutions of Problem $*$

A hypersurface $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ is called *holonomic* if $M^n$ carries global orthogonal coordinates $u_1, \ldots, u_n$ such that the coordinate vector fields $\partial_j = \dfrac{\partial}{\partial u_j}$ are everywhere eigenvectors of the shape operator $A$ of $f$. Denote $v_j = \|\partial_j\|$, and let $V_j \in C^\infty(M)$, $1 \leq j \leq n$, be defined by $A\partial_j = v_j^{-1} V_j \partial_j$. The first and second fundamental forms of $f$ are then given by

$$I = \sum_{i=1}^{n} v_i^2 du_i^2 \quad \text{and} \quad II = \sum_{i=1}^{n} V_i v_i du_i^2. \tag{1}$$

Set $v = (v_1, \ldots, v_n)$ and $V = (V_1, \ldots, V_n)$. We call $(v, V)$ the pair associated to $f$. In the next well-known result and in the sequel, for $s \in \{0, 1\}$ we denote $\epsilon_s = -2s + 1$.

**Proposition 1** *The triple $(v, h, V)$, where $h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i}$, satisfies the system of PDE's*

$$\begin{cases} \text{(i)}\ \dfrac{\partial v_i}{\partial u_j} = h_{ji} v_j, \qquad \text{(ii)}\ \dfrac{\partial h_{ik}}{\partial u_j} = h_{ij} h_{jk}, \\ \text{(iii)}\ \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + h_{ki} h_{kj} + \epsilon_s V_i V_j + c v_i v_j = 0, \\ \text{(iv)}\ \dfrac{\partial V_i}{\partial u_j} = h_{ji} V_j, \quad 1 \le i \ne j \ne k \ne i \le n. \end{cases} \tag{2}$$

*Conversely, if $(v, h, V)$ is a solution of (2) on a simply connected open subset $U \subset \mathbb{R}^n$, with $v_i \ne 0$ everywhere for all $1 \le i \le n$, then there exists a holonomic hypersurface $f : U \to \mathbb{Q}_s^{n+1}(c)$ whose first and second fundamental forms are given by (1).*

Conformally flat hypersurfaces $f : M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures have been characterized in Canevari and Tojeiro (2017) as follows.

**Theorem 2** *Let $f : M^3 \to \mathbb{Q}_s^4(c)$ be a holonomic hypersurface whose associated pair $(v, V)$ satisfies*

$$\sum_{i=1}^{3} \delta_i v_i^2 = 0, \quad \sum_{i=1}^{3} \delta_i v_i V_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = 1, \tag{3}$$

*where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. Then $M^3$ is conformally flat and $f$ has three distinct principal curvatures.*

*Conversely, any conformally flat hypersurface $f : M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures is locally a holonomic hypersurface whose associated pair $(v, V)$ satisfies (3).*

The characterization in Canevari and Tojeiro (2017) of the solutions $f : M^3 \to \mathbb{Q}_s^4(c)$ of Problem $*$ with three distinct principal curvatures is as follows.

**Theorem 3** *Let $f : M^3 \to \mathbb{Q}_s^4(c)$ be a simply connected holonomic hypersurface whose associated pair $(v, V)$ satisfies*

$$\sum_{i=1}^{3} \delta_i v_i^2 = \hat{\epsilon}, \quad \sum_{i=1}^{3} \delta_i v_i V_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = C := \tilde{\epsilon}(c - \tilde{c}), \tag{4}$$

*where $\hat{\epsilon}, \tilde{\epsilon} \in \{-1, 1\}$, $\tilde{c} \ne c$, $\hat{\epsilon}\tilde{\epsilon} = \epsilon_s$, $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ either if $\hat{\epsilon} = 1$ or if $\hat{\epsilon} = -1$ and $C > 0$, and $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$ if $\hat{\epsilon} = -1$ and $C < 0$. Then $M^3$ admits an isometric immersion into $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, with $\epsilon_{\tilde{s}} = \tilde{\epsilon}$, which is unique up to congruence.*

*Conversely, if $f : M^3 \to \mathbb{Q}_s^4(c)$ is a hypersurface with three distinct principal curvatures for which there exists an isometric immersion $\tilde{f} : M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $\tilde{c} \ne c$, then $f$ is locally a holonomic hypersurface whose associated pair $(v, V)$ satisfies (4), with $\tilde{\epsilon} = \epsilon_{\tilde{s}}$.*

## 3 The Ribaucour Transformation

Two immersions $f\colon M^n \to \mathbb{R}_s^{n+p}$ and $f'\colon M^n \to \mathbb{R}_s^{n+p}$ are said to be related by a Ribaucour transformation if $|f - f'| \neq 0$ everywhere and there exist a vector bundle isometry $\mathcal{P}\colon f^*T\mathbb{R}_s^{n+p} \to f'^*T\mathbb{R}_s^{n+p}$, a tensor $D \in \Gamma(T^*M \otimes TM)$, which is symmetric with respect to the induced metric, and a nowhere vanishing $\delta \in \Gamma(f^*T\mathbb{R}_s^{n+p})$ such that

(a) $\mathcal{P}(Z) - Z = \langle \delta, Z\rangle (f - f')$ for all $Z \in \Gamma(f^*T\mathbb{R}_s^{n+p})$;
(b) $\mathcal{P} \circ f_* \circ D = f'_*$.

Given an immersion $f\colon M^n \to \mathbb{Q}_s^{n+p}(c)$, with $c \neq 0$, let $F = i \circ f\colon M^n \to \mathbb{R}_{s+\epsilon_0}^{n+p+1}$, where $\epsilon_0 = 0$ or $1$ corresponding to $c > 0$ or $c < 0$, respectively, and $i\colon \mathbb{Q}_s^{n+p}(c) \to \mathbb{R}_{s+\epsilon_0}^{n+p+1}$ denotes an umbilical inclusion. An immersion $f'\colon M^n \to \mathbb{Q}_s^{n+p}(c)$ is said to be a Ribaucour transform of $f$ with data $(\mathcal{P}, D, \delta)$ if $F' = i \circ f'\colon M^n \to \mathbb{R}_{s+\epsilon_0}^{n+p+1}$ is a Ribaucour transform of $F$ with data $(\hat{\mathcal{P}}, D, \hat{\delta})$, where $\hat{\delta} = \delta - cF$ and $\hat{\mathcal{P}}\colon F^*T\mathbb{R}_{s+\epsilon_0}^{n+p+1} \to F'^*T\mathbb{R}_{s+\epsilon_0}^{n+p+1}$ is the extension of $\mathcal{P}$ such that $\hat{\mathcal{P}}(F) = F'$. The next result was proved in Dajczer and Tojeiro (2003).

**Theorem 4** *Let $f\colon M^n \to \mathbb{Q}_s^{n+p}(c)$ be an isometric immersion of a simply connected Riemannian manifold and let $f'\colon M^n \to \mathbb{Q}_s^{n+p}(c)$ be a Ribaucour transform of $f$ with data $(\mathcal{P}, D, \delta)$. Then there exist $\varphi \in C^\infty(M)$ and $\hat{\beta} \in \Gamma(N_f M)$ satisfying*

$$\alpha_f(\nabla\varphi, X) + \nabla_X^\perp \hat{\beta} = 0 \quad \textit{for all} \quad X \in TM \tag{5}$$

*such that $F' = i \circ f'$ and $F = i \circ f$ are related by*

$$F' = F - 2\nu\varphi\mathcal{G}, \tag{6}$$

*where $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ and $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$. Moreover,*

$$\hat{\mathcal{P}} = I - 2\nu\mathcal{G}\mathcal{G}^*, \quad D = I - 2\nu\varphi\Phi \quad \textit{and} \quad \hat{\delta} = -\varphi^{-1}\mathcal{G}, \tag{7}$$

*where $\Phi = Hess\, \varphi + c\varphi I - A^f_{\hat{\beta}}$.*

*Conversely, given $\varphi \in C^\infty(M)$ and $\hat{\beta} \in \Gamma(N_f M)$ satisfying (5) such that $\varphi\nu \neq 0$ everywhere, let $U \subset M^n$ be an open subset where the tensor $D$ given by (7) is invertible, and let $F'\colon U \to \mathbb{R}_{s+\epsilon_o}^{n+p+1}$ be defined by (6). Then $F' = i \circ f'$, where $f'$ is a Ribaucour transform of $f$. Moreover, the second fundamental forms of $f$ and $f'$ are related by*

$$\tilde{A}^{f'}_{\mathcal{P}\xi} = D^{-1}(A^f_\xi + 2\nu\langle \hat{\beta}, \xi\rangle \Phi) \tag{8}$$

*for all $\xi \in \Gamma(N_f M)$.*

We now derive from Theorem 4 a Ribaucour transformation for holonomic hypersurfaces, in a form that is slightly different from the one in Dajczer and Tojeiro (2002). For that we need the following.

**Proposition 5** *Let $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ be a holonomic hypersurface with associated pair $(v, V)$. Then, the linear system of PDE's*

$$\begin{cases} \text{(i)}\ \dfrac{\partial \varphi}{\partial u_i} = v_i \gamma_i, \quad \text{(ii)}\ \dfrac{\partial \gamma_j}{\partial u_i} = h_{ji}\gamma_i, \quad i \neq j, \\ \text{(iii)}\ \dfrac{\partial \gamma_i}{\partial u_i} = (v_i - v_i')\psi - \sum_{j\neq i} h_{ji}\gamma_j + \beta V_i - c\varphi v_i, \\ \text{(iv)}\ \epsilon_s \dfrac{\partial \beta}{\partial u_i} = -V_i\gamma_i, \\ \text{(v)}\ \dfrac{\partial \log \psi}{\partial u_i} = -\dfrac{\gamma_i v_i'}{\varphi}, \quad \text{(vi)}\ \dfrac{\partial v_i'}{\partial u_j} = h_{ji}' v_j', \quad i \neq j, \end{cases} \tag{9}$$

*with $h_{ij}$ and $h'_{ij}$ given by*

$$h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i} \quad \textit{and} \quad h'_{ij} = h_{ij} + (v_j' - v_j)\frac{\gamma_i}{\varphi}, \tag{10}$$

*is completely integrable and has the first integral*

$$\sum_i {\gamma_i}^2 + \epsilon_s\beta^2 + c\varphi^2 - 2\varphi\psi = K \in \mathbb{R}. \tag{11}$$

*Proof* A straightforward computation. □

**Theorem 6** *Let $f: M^n \to \mathbb{Q}_s^{n+1}(c)$ be a holonomic hypersurface with associated pair $(v, V)$. If $f': M^n \to \mathbb{Q}_s^{n+1}(c)$ is a Ribaucour transform of $f$, then there exists a solution $(\gamma, v', \varphi, \psi, \beta)$ of (9) satisfying*

$$\sum_i {\gamma_i}^2 + \epsilon_s\beta^2 + c\varphi^2 - 2\varphi\psi = 0 \tag{12}$$

*such that $F' = i \circ f'$ and $F = i \circ f$ are related by*

$$F' = F - \frac{1}{\psi}\left(\sum_i \gamma_i F_* e_i + \beta i_*\xi + c\varphi F\right), \tag{13}$$

*where $\xi$ is a unit normal vector field to $f$ and $e_i = v_i^{-1}\partial_i$, $1 \leq i \leq n$.*

*Conversely, given a solution $(\gamma, v', \varphi, \psi, \beta)$ of (9) satisfying (12) on an open subset $U \subset M^n$ where $v_i'$ is positive for $1 \leq i \leq n$, then $F'$ defined by (13) is an immersion such that $F' = i \circ f'$, where $f'$ is a Ribaucour transform of $f$ whose associated pair is $(v', V')$, with*

$$V_i' = V_i + (v_i - v_i')\frac{\epsilon_s\beta}{\varphi}, \quad 1 \leq i \leq n. \tag{14}$$

*Proof* Let $f': M^n \to \mathbb{Q}_s^{n+1}(c)$ be a Ribaucour transform of $f$. By Theorem 4, there exist $\varphi \in C^\infty(M)$ and $\hat{\beta} \in \Gamma(N_f M)$ satisfying (5) such that $F'$ is given by (6), where $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ and $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$.

Write $\nabla\varphi = \sum_{i=1}^{n} \gamma_i e_i$, where $\gamma_i \in C^\infty(M)$, $1 \le i \le n$. Since $\partial_i = v_i e_i$, $1 \le i \le n$, this is equivalent to equation (i) of system (9). Now write $\hat{\beta} = \beta\xi$, where $\beta \in C^\infty(M)$. Then (5) can be written as

$$A\nabla\varphi = -\epsilon_s \nabla\beta, \tag{15}$$

which is equivalent, by taking inner products of both sides with $\partial_i$, to equation (iv) of system (9). On the other hand, Eq. (5) implies that

$$\mathcal{G}_* = F_*\Phi,$$

where $\Phi = \text{Hess}\,\varphi + c\varphi I - A^f_{\hat{\beta}}$. Therefore $\Phi$ is a Codazzi tensor that satisfies

$$\alpha_f(\Phi X, Y) = \alpha_f(X, \Phi Y)$$

for all $X, Y \in TM$, that is, $\Phi$ has $\{e_1, \ldots, e_n\}$ as a diagonalyzing frame. Since

$$\Phi\partial_i = \left(\frac{\partial\gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + cv_i\varphi\right) e_i + \sum_{j\neq i}\left(\frac{\partial\gamma_j}{\partial u_i} - h_{ji}\gamma_i\right) e_j, \tag{16}$$

equation (ii) of system (9) follows. Now define $\psi \in C^\infty(M)$ by

$$2\varphi\psi = \langle\mathcal{G}, \mathcal{G}\rangle = \sum_i {\gamma_i}^2 + \epsilon_s\beta^2 + c\varphi^2.$$

Differentiating both sides with respect to $u_i$ and using (i), (ii) and (iv) of (9) yields

$$\frac{\partial\gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + cv_i\varphi = v_i\psi + \frac{\varphi}{\gamma_i}\frac{\partial\psi}{\partial u_i}. \tag{17}$$

Defining $v_i'$ by (v), then (iii) follows from (17). Finally, from

$$\frac{\partial^2\gamma_i}{\partial u_i\partial u_j} = \frac{\partial^2\gamma_i}{\partial u_j\partial u_i}$$

we obtain

$$\frac{\partial}{\partial u_i}\left(h_{ij}\gamma_j\right) = \frac{\partial}{\partial u_j}\left((v_i - v_i')\psi - \sum_{k\neq i} h_{ki}\gamma_k + \beta V_i - c\varphi v_i\right),$$

thus

$$\frac{\partial h_{ij}}{\partial u_i}\gamma_j + h_{ij}\frac{\partial \gamma_j}{\partial u_i} = \left(\frac{\partial v_i}{\partial u_j} - \frac{\partial v_i'}{\partial u_j}\right)\psi + (v_i - v_i')\frac{\partial \psi}{\partial u_j} - \frac{\partial h_{ji}}{\partial u_j}\gamma_j - h_{ji}\frac{\partial \gamma_j}{\partial u_j}$$
$$- \frac{\partial h_{ki}}{\partial u_j}\gamma_k - h_{ki}\frac{\partial \gamma_k}{\partial u_j} + \frac{\partial \beta}{\partial u_j}V_i + \beta\frac{\partial V_i}{\partial u_j} - c\frac{\partial \varphi}{\partial u_j}v_i - c\varphi\frac{\partial v_i}{\partial u_j}.$$

It follows that

$$\left(\frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ji}}{\partial u_j}\right)\gamma_j + h_{ij}h_{ji}\gamma_i = \psi h_{ji}v_j - \frac{\partial v_i'}{\partial u_j}\psi - (v_i - v_i')\frac{\gamma_j\psi}{\varphi}v_j'$$
$$- h_{ji}(v_j - v_j')\psi + h_{ji}h_{ij}\gamma_i + h_{ji}h_{kj}\gamma_k - \beta h_{ji}V_j + c\varphi h_{ji}v_j - h_{kj}h_{ji}\gamma_k$$
$$- h_{ki}h_{kj}\gamma_j - V_iV_j\gamma_j + \beta h_{ji}V_j - cv_iv_j\gamma_j - c\varphi h_{ji}v_j,$$

which yields equation (vi) of (9).

Conversely, let $F'$ be given by (13) in terms of a solution $(\gamma, v', \varphi, \psi, \beta)$ of (9) satisfying (12) on an open subset $U \subset M^n$ where $v_i'$ is nowhere vanishing for $1 \leq i \leq n$. We have $\nabla\varphi = \sum_{i=1}^n \gamma_i e_i$ by equation (i) of (9). Defining $\hat{\beta} \in \Gamma(N_f M)$ by $\hat{\beta} = \beta\xi$, we can write $F'$ as in (6), with $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ and $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$. In view of (iv), Eq. (15) is satisfied, and hence so is (5). Thus $\mathcal{G}_* = F_* \circ \Phi$, where $\Phi = \text{Hess}\ \varphi + c\varphi I - A^f_{\hat{\beta}}$.

It follows from (ii) and (16) that $\Phi\partial_i = B_i\partial_i$, where

$$B_i = v_i^{-1}\left(\frac{\partial \gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + cv_i\varphi\right) = v_i^{-1}(v_i - v_i')\psi.$$

Using (iii) and (12) we obtain

$$D\partial_i = (1 - 2\nu\varphi B_i)\partial_i = (1 - 2\nu\varphi v_i^{-1}(v_i - v_i')\psi) = \frac{v_i'}{v_i}\partial_i.$$

Thus $D$ is invertible wherever $v_i'$ does not vanish for $1 \leq i \leq n$. It follows from Theorem 4 that the map $F'$ defined by (13) is an immersion on $U$ and that $F' = i \circ f'$, where $f'$ is a Ribaucour transform of $f$. Moreover, we obtain from (8) that $F'$, and hence $f'$, is holonomic with $u_1, \ldots, u_n$ as principal coordinates. It also follows from (8) that

$$\frac{V_i'}{v_i'}\partial_i = A^{f'}\partial_i = \frac{v_i}{v_i'}\left(\frac{V_i}{v_i} + \frac{\epsilon_s\beta}{\varphi}\frac{v_i - v_i'}{v_i}\right)\partial_i,$$

which yields (14). $\square$

## 4 The Transformation of Conformally Flat Hypersurfaces and Solutions of Problem $*$

We now specialize the Ribaucour transformation to the classes of hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ that are either conformally flat or admit an isometric immersion into $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $\tilde{c} \neq c$.

**Proposition 7** *If $f\colon M^3 \to \mathbb{Q}_s^4(c)$ is a holonomic hypersurface whose associated pair $(v, V)$ satisfies* (4) (*respectively,* (3)), *then the linear system of PDE's obtained by adding the equation*

$$\delta_i \frac{\partial v_i'}{\partial u_i} + \delta_j h_{ij}' v_j' + \delta_k h_{ik}' v_k' = 0 \tag{18}$$

*to system* (9), *where $h_{ij}'$ is given by* (10), *is completely integrable and has* (*besides* (11)) *the first integral*

$$\delta_1 v_1'^2 + \delta_2 v_2'^2 + \delta_3 v_3'^2 = K \in \mathbb{R}. \tag{19}$$

*Moreover, the function*

$$\Omega = \varphi \sum_{j=1}^{3} \delta_j v_j' V_j - \epsilon_s \beta \left( K - \sum_{j=1}^{3} \delta_j v_j v_j' \right) \tag{20}$$

*satisfies*

$$\frac{\partial \Omega}{\partial u_i} = \frac{\gamma_i}{\varphi}(v_i + v_i')\Omega. \tag{21}$$

*In particular, if initial conditions for $\varphi$ and $\beta$ at $x_0 \in M^3$ are chosen so that $\Omega$ vanishes at $x_0$, then $\Omega$ vanishes everywhere.*

*Proof* The first two assertions follow from straightforward computations. To prove the last one, define $\rho = \sum_{i=1}^{3} \delta_i v_i' V_i$ and $\Theta = K - \sum_{i=1}^{3} \delta_i v_i' v_i$. We have

$$\begin{aligned}
\frac{\partial \rho}{\partial u_i} &= \delta_i \frac{\partial v_i'}{\partial u_i} V_i + \delta_i v_i' \frac{\partial V_i}{\partial u_i} + \sum_{j \neq i} \delta_j \frac{\partial v_j'}{\partial u_i} V_j + \sum_{j \neq i} \delta_j v_j' \frac{\partial V_j}{\partial u_i} \\
&= \sum_{j \neq i} \delta_j (h_{ij} - h_{ij}') v_j' V_i - \sum_{j \neq i} \delta_j (h_{ij} - h_{ij}') V_j v_i' \\
&= \sum_{j \neq i} \delta_j (v_j' - v_j) V_j \frac{\gamma_i v_i'}{\varphi} - \sum_{j \neq i} \delta_j (v_j' - v_j) v_j' \frac{\gamma_i V_i}{\varphi} \\
&= \frac{v_i' \gamma_i}{\varphi} \left( \rho - \delta_i v_i' V_i + \delta_i v_i V_i \right) - \frac{V_i \gamma_i}{\varphi} \left( \Theta - \delta_i v_i'^2 + \delta_i v_i v_i' \right) \\
&= \frac{\gamma_i}{\varphi} (v_i' \rho - \Theta V_i)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \Theta}{\partial u_i} &= -\delta_i \frac{\partial v_i}{\partial u_i} v_i' - \delta_i v_i \frac{\partial v_i'}{\partial u_i} - \sum_{j\neq i} \delta_j \frac{\partial v_j}{\partial u_i} v_j' - \sum_{j\neq i} \delta_j v_j \frac{\partial v_j'}{\partial u_i} \\
&= \left(\sum_{j\neq i} \delta_j v_j (h_{ij} - h_{ij}')\right) v_i' + \left(\sum_{j\neq i} \delta_j (h_{ij}' - h_{ij}) v_j'\right) v_i \\
&= \left(\sum_{j\neq i} \delta_j v_j (v_j - v_j')\right) \frac{\gamma_i v_i'}{\varphi} + \left(\sum_{j\neq i} \delta_j (v_j' - v_j) v_j'\right) \frac{v_i \gamma_i}{\varphi} \\
&= (\Theta - \delta_i v_i^2 + \delta_i v_i v_i'))\frac{\gamma_i v_i'}{\varphi} + (\Theta - \delta_i v_i'^2 + \delta_i v_i v_i'))\frac{v_i \gamma_i}{\varphi} \\
&= \frac{\gamma_i}{\varphi}(v_i + v_i')\Theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \Omega}{\partial u_i} &= \frac{\partial \varphi}{\partial u_i}\rho + \varphi \frac{\partial \rho}{\partial u_i} - \frac{\partial \epsilon_s \beta}{\partial u_i}\Theta - \epsilon_s \beta \frac{\partial \Theta}{\partial u_i} \\
&= v_i \gamma_i \rho + \varphi \frac{\gamma_i}{\varphi}(v_i'\rho - \Theta V_i) + V_i \gamma_i \Theta - \epsilon_s \beta \frac{\gamma_i}{\varphi}(v_i + v_i')\Theta \\
&= \rho \gamma_i (v_i + v_i') - \frac{\epsilon_s \beta \gamma_i}{\varphi}(v_i + v_i')\Theta \\
&= \frac{\gamma_i}{\varphi}(v_i + v_i')\Omega,
\end{aligned}$$

which proves (21). The last assertion follows from (21) and the lemma below. □

**Lemma 8** *Let $M^n$ be a connected manifold and let $\Omega \in C^\infty(M)$. Assume that there exists a smooth one-form $\omega$ on $M^n$ such that $d\Omega = \omega\Omega$. If $\Omega$ vanishes at some point of $M^n$, then it vanishes everywhere.*

*Proof* Given any smooth curve $\gamma\colon I \to M^n$ with $0 \in I$, denote $\lambda(s) = \omega(\gamma'(s))$. By the assumption we have

$$(\Omega \circ \gamma)(t) = (\Omega \circ \gamma)(0) \exp \int_0^t \lambda(s) ds,$$

and the conclusion follows from the connectedness of $M^n$. □

**Theorem 9** *Let $f\colon M^3 \to \mathbb{Q}_s^4(c)$ be a holonomic hypersurface whose associated pair $(v, V)$ satisfies (4) (respectively, (3)) and $f'\colon M^3 \to \mathbb{Q}_s^4(c)$ a Ribaucour transform of $f$ determined by a solution $(\gamma_1, \gamma_2, \gamma_3, v_1', v_2', v_3', \varphi, \psi, \beta)$ of system (9). If the* *associated pair* $(v', V')$ *of* $f'$ *also satisfies* (4) (*respectively,* (3))*, then*

$$\Omega := \varphi \sum_{j=1}^{3} \delta_j v'_j V_j - \epsilon_s \beta \left( K - \sum_{j=1}^{3} \delta_j v_j v'_j \right) = 0, \tag{22}$$

*with* $K = \hat{\epsilon}$ (*respectively,* $K = 0$). *Conversely, let* $(\gamma_1, \gamma_2, \gamma_3, v'_1, v'_2, v'_3, \varphi, \psi, \beta)$ *be a solution of the linear system of PDE's obtained by adding Eq.* (18) *to system* (9)*. If* (12)*,* (22) *and*

$$\sum_{i=1}^{3} \delta_i {v'_i}^2 = K, \tag{23}$$

*where* $K = \hat{\epsilon}$ (*respectively,* $K = 0$)*, are satisfied at some point of* $M^3$*, then (they are satisfied at every point of* $M^3$ *and) the pair* $(v', V')$ *associated to the Ribaucour transform of* $f$ *determined by such a solution also satisfies* (4) (*respectively,* (3)).

*Proof* Let $(v', V')$ be the pair associated to $f'$. Then, using conditions (4) (respectively, (3)), we obtain

$$\begin{aligned}
\sum_{j=1}^{3} \delta_j {V'_j}^2 - \sum_{j=1}^{3} \delta_j {V_j}^2 &= \sum_{j=1}^{3} \delta_j (V'_j - V_j)(V'_j + V_j) \\
&= \frac{\epsilon_s \beta}{\varphi} \sum_{j=1}^{3} \delta_j (v_j - v'_j) \left( 2V_j + \frac{\epsilon_s \beta}{\varphi} (v_j - v'_j) \right) \\
&= \frac{\epsilon_s \beta}{\varphi} \left( 2 \sum_{j=1}^{3} \delta_j V_j (v_j - v'_j) + \frac{\epsilon_s \beta}{\varphi} \sum_{j=1}^{3} \delta_j (v_j - v'_j)^2 \right) \\
&= \frac{\epsilon_s \beta}{\varphi^2} \left( -2\Omega + \epsilon_s \beta \left( \sum_{j=1}^{3} \delta_j {v'_j}^2 - K \right) \right),
\end{aligned} \tag{24}$$

where $K = \hat{\epsilon}$ (respectively, $K = 0$). If the pair $(v', V')$ associated to $f'$ satisfies (4) (respectively, (3)), then (23) holds, as well as

$$\sum_{j=1}^{3} \delta_j v'_j V'_j = 0 \tag{25}$$

and

$$\sum_{j=1}^{3} \delta_j {V'_j}^2 = C, \tag{26}$$

where $C = \tilde{\epsilon}(c - \tilde{c})$ (respectively, $C = 1$). It follows from (24) that (22) holds.

Conversely, let $(\gamma_1, \gamma_2, \gamma_3, v'_1, v'_2, v'_3, \varphi, \psi, \beta)$ be a solution of the linear system of PDE's obtained by adding Eq. (18) to system (9). If (12), (23) and (22) are satisfied at some point of $M^n$, then they are satisfied at every point of $M^n$ by Proposition 7. Then, Eqs. (23), (22) and (24) imply that (26) holds. On the other hand, using (14) we obtain

$$\begin{aligned}\sum_{j=1}^{3}\delta_j v'_j V'_j &= \sum_{j=1}^{3}\delta_j v'_j V_j + \frac{\epsilon_s\beta}{\varphi}\sum_{j=1}^{3}\delta_j v'_j v_j - \frac{\epsilon_s\beta}{\varphi}\sum_{j=1}^{3}\delta_j {v'_j}^2\\ &= \sum_{j=1}^{3}\delta_j v'_j V_j - \frac{\epsilon_s\beta}{\varphi}(K - \sum_{j=1}^{3}\delta_j v'_j v_j)\\ &= \varphi^{-1}\Omega = 0\end{aligned}$$

by (23) and (22). Thus the pair $(v', V')$ associated to $f'$ also satisfies (4) (respectively, (3)). $\square$

### 4.1 Explicit Solutions of Problem $*$

We now use Theorem 9 to compute explicit examples of pairs of isometric immersions $f: M^3 \to \mathbb{Q}_s^4(c)$ and $\tilde{f}: M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, $c \neq \tilde{c}$, with three distinct principal curvatures.

First notice that, if $c = 0$ (respectively, $c \neq 0$) and $(v, h, V)$ is a solution of system (2) on a simply connected open subset $U \subset \mathbb{R}^3$ with $v_i \neq 0$ everywhere for $1 \leq i \leq 3$, then, in order to determine the corresponding immersion $f: U \to \mathbb{R}_s^4$ (respectively, $f: U \to \mathbb{Q}_s^4(c) \subset \mathbb{R}_{s+\epsilon_0}^5$, where $\epsilon_0 = c/|c|$), one has to integrate the system of PDE's

$$\begin{cases}\text{(i)}\ \ \dfrac{\partial f}{\partial u_i} = v_i X_i, \quad \text{(ii)}\ \ \dfrac{\partial X_i}{\partial u_j} = h_{ij} X_j, \ \ i \neq j,\\ \text{(iii)}\ \ \dfrac{\partial X_i}{\partial u_i} = -\sum_{k\neq i} h_{ki} X_k + \epsilon_s V_i N - c v_i f,\\ \text{(iv)}\ \ \dfrac{\partial N}{\partial u_i} = -V_i X_i, \ \ 1 \leq i \leq 3,\end{cases} \tag{27}$$

with initial conditions $X_1(u_0), X_2(u_0), X_3(u_0), N(u_0), f(u_0)$ at some point $u_0 \in U$ chosen so that the set $\{X_1(u_0), X_2(u_0), X_3(u_0), N(u_0)\}$ (respectively, $\{X_1(u_0), X_2(u_0), X_3(u_0), N(u_0), |c|^{1/2} f(u_0)\}$) is an orthonormal basis of $\mathbb{R}_s^4$ (respectively, $\mathbb{R}_{s+\epsilon_0}^5$).

The idea for the construction of explicit examples is to start with trivial solutions $(v, h, V)$ of system (2). If $\hat{\epsilon} = 1$, one can start with the solution $(v, h, V)$ of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (1, 0, 0)$, $h = 0$ and $V$ is either $\sqrt{-C}(0, 1, 0)$ or $\sqrt{C}(0, 0, 1)$, corresponding to $C < 0$ or $C > 0$, respectively. If $\hat{\epsilon} = -1$ and $C > 0$, we may start with the solution $(v, h, V)$ of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (0, 1, 0)$, $h = 0$ and $V = \sqrt{C}(0, 0, 1)$, whereas for $C < 0$ we take $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$, $v = (0, 0, 1)$, $h = 0$ and $V = \sqrt{-C}(1, 0, 0)$. Even though, for the corresponding solution $(X_1, X_2, X_3, N, f)$ of system (27), the map $f: U \to \mathbb{Q}_s^4(c)$ is not an immersion, the map $f': U \to \mathbb{Q}_s^4(c)$

obtained by applying Theorem 9 to it does define a hypersurface of $\mathbb{Q}_s^4(c)$, which is therefore a solution of Problem $*$.

In the following, we consider the case in which $\hat{\epsilon} = 1$ and $C < 0$, the others being similar. We take $(v, h, V)$ as the solution of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (1, 0, 0)$, $h = 0$ and $V = \sqrt{-C}(0, 1, 0)$.

If $c = 0$, the corresponding solution of system (27) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, \epsilon E_4, 0)$$

is given by

$$\begin{aligned} f &= f(u_1) = u_1 E_1, \quad X_1 = E_1, \quad X_3 = E_3, \\ X_2 &= \begin{cases} \cosh au_2 E_2 + \sinh au_2 E_4, & \text{if } \epsilon_s = -1, \\ \cos au_2 E_2 + \sin au_2 E_4, & \text{if } \epsilon_s = 1, \end{cases} \end{aligned} \tag{28}$$

and

$$N = \begin{cases} -\sinh au_2 E_2 - \cosh au_2 E_4, & \text{if } \epsilon_s = -1, \\ -\sin au_2 E_2 + \cos au_2 E_4, & \text{if } \epsilon_s = 1, \end{cases} \tag{29}$$

where $a = \sqrt{-C}$. If $c \neq 0$, the corresponding solution of system (27) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, E_4, |c|^{-1/2} E_5)$$

is given by

$$f = f(u_1) = \begin{cases} \frac{1}{\sqrt{c}}(\cos\sqrt{c}\, u_1 E_5 + \sin\sqrt{c}\, u_1 E_1), & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}}(\cosh\sqrt{-c}\, u_1 E_5 + \sinh\sqrt{-c}\, u_1 E_1), & \text{if } c < 0, \end{cases} \tag{30}$$

$$X_1 = \begin{cases} -\sin\sqrt{c}\, u_1 E_5 + \cos\sqrt{c}\, u_1 E_1, & \text{if } c > 0, \\ \sinh\sqrt{-c}\, u_1 E_5 + \cosh\sqrt{-c}\, u_1 E_1, & \text{if } c < 0, \end{cases} \tag{31}$$

$X_3 = E_3$ and $X_2$, $N$ as in (28) and (29), respectively.

We now solve system (9) for $(v, h, V)$ as in the preceding paragraph. Notice that (11) and (19), with $K = 0$ (respectively, $K = 1$) in (11) (respectively, (19)), reduce, respectively, to

$$2\varphi\psi = \sum_i \gamma_i^2 + \epsilon_s \beta^2 + c\varphi^2 \tag{32}$$

and

$$v_2'^2 = v_1'^2 + v_3'^2 - 1. \tag{33}$$

We also impose that

$$-a\varphi v_2' = \epsilon_s \beta (1 - v_1'), \tag{34}$$

which corresponds to the function $\Omega$ in (22) vanishing everywhere. It follows from equations (i), (ii) and (iv) of system (9) that $\varphi$, $\gamma_j$ and $\beta$ depend only on $u_1$, $u_j$ and $u_2$, respectively. Equation (iii) then implies that there exist smooth functions $\phi_i = \phi_i(u_i)$, $1 \leq i \leq 3$, such that

$$(\delta_{1i} - v_i')\psi = \phi_i. \tag{35}$$

Replacing (35) in (33) gives

$$\psi = \frac{\phi_1^2 - \phi_2^2 + \phi_3^2}{2\phi_1}. \tag{36}$$

Multiplying (34) by $\psi$ and using (35) yields

$$a\varphi\phi_2 = \epsilon_s\beta\phi_1,$$

hence there exists $K \neq 0$ such that

$$\beta = \frac{\epsilon_s}{K}\phi_2 \quad \text{and} \quad \varphi = \frac{1}{Ka}\phi_1. \tag{37}$$

It follows from (i) and (iv) that

$$\gamma_1 = \frac{1}{Ka}\phi_1' \quad \text{and} \quad \gamma_2 = -\frac{1}{Ka}\phi_2' \tag{38}$$

where $\phi_i'$ stands for the derivative of $\phi_i$ (with respect to $u_i$). Using (v) for $i = 3$, (35) and the second equation in (37) we obtain that

$$\gamma_3 = \frac{1}{Ka}\phi_3'.$$

Then, it follows from (iii), (35), the first equation in (38) and the second one in (37) that

$$\phi_1'' = (Ka - c)\phi_1. \tag{39}$$

Similarly,

$$\phi_2'' = -(\epsilon_s a^2 + Ka)\phi_2 \quad \text{and} \quad \phi_3'' = Ka\phi_3. \tag{40}$$

Moreover, by (32) we must have

$$(\phi_1'^2 - (Ka - c)\phi_1^2) + (\phi_2'^2 + (\epsilon_s a^2 + Ka)\phi_2^2) + (\phi_3'^2 - Ka\phi_3^2) = 0. \tag{41}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (39) and (40).

We compute explicitly the corresponding hypersurface given by (13) when $c = 0$, $\tilde{c} = 1$, $\epsilon_s = 1 = \tilde{\epsilon}$ and $K = 1$. In this case we have $C = -1$ and $a = 1$, hence

Eqs. (39) and (40) yield

$$\begin{cases} \phi_1 = A_{11}\cosh u_1 + A_{12}\sinh u_1, \\ \phi_2 = A_{21}\cos\sqrt{2}\,u_2 + A_{22}\sin\sqrt{2}\,u_2, \\ \phi_3 = A_{31}\cosh u_3 + A_{32}\sinh u_3, \end{cases}$$

where $A_{ij} \in \mathbb{R}$, $1 \le i, j \le 3$, satisfy

$$A_{12}^2 - A_{11}^2 + 2(A_{21}^2 + A_{22}^2) + A_{32}^2 - A_{31}^2 = 0,$$

in view of (41). Assuming, say, that

$$A_{12}^2 - A_{11}^2 < 0 \quad \text{and} \quad A_{32}^2 - A_{31}^2 < 0,$$

we may write $A_{11} = \rho_1\cosh\theta_1$, $A_{12} = \rho_1\sinh\theta_1$, $A_{21} = \rho_2\sin\theta_2$, $A_{22} = \rho_2\cos\theta_2$, $A_{31} = \rho_3\cosh\theta_3$ and $A_{32} = \rho_3\sinh\theta_3$ for some $\rho_i > 0$ and $\theta_i \in \mathbb{R}$, $1 \le i \le 3$. Then

$$\begin{cases} \phi_1 = \rho_1\cosh(u_1 + \theta_1), \\ \phi_2 = \rho_2\sin(\sqrt{2}\,u_2 + \theta_2), \\ \phi_3 = \rho_3\cosh(u_3 + \theta_3), \end{cases}$$

with

$$2\rho_2^2 = \rho_1^2 + \rho_3^2,$$

and we can assume that $\theta_i = 0$ after a suitable change $u_i \mapsto u_i + u_i^0$ of the coordinates $u_i$, $1 \le i \le 3$. Setting $\rho = \rho_2$, we can write $\rho_1 = \sqrt{2}\rho\cos\theta$ and $\rho_3 = \sqrt{2}\rho\sin\theta$ for some $\theta \in [0, 2\pi]$. Thus

$$\begin{cases} \phi_1 = \sqrt{2}\rho\cos\theta\cosh u_1, \\ \phi_2 = \rho\sin\sqrt{2}\,u_2, \\ \phi_3 = \sqrt{2}\rho\sin\theta\cosh u_3, \end{cases}$$

and the coordinate functions of the corresponding one-parameter family (with $\theta$ as the parameter) of hypersurfaces $f' = f'_\theta \colon U \to \mathbb{R}^4$ are

$$f'_1 = u_1 - 2gh\cos\theta\sinh u_1, \quad f'_2 = gh(2\cos\sqrt{2}u_2\cos u_2 + \sqrt{2}\sin\sqrt{2}u_2\sin u_2),$$
$$f'_3 = -2gh\sin\theta\sinh u_3, \quad f'_4 = gh(2\cos\sqrt{2}u_2\sin u_2 - \sqrt{2}\sin\sqrt{2}u_2\cos u_2),$$

where

$$g = 2\cos\theta\cosh u_1 \tag{42}$$

and

$$h^{-1} = 2\cos^2\theta\cosh^2 u_1 - \sin^2\sqrt{2}u_2 + 2\sin^2\theta\cosh^2 u_3. \tag{43}$$

To determine the immersion $\tilde{f}': U \to \mathbb{S}^4$ that has the same induced metric as $f'$, we start with the solution $(\tilde{v}, \tilde{h}, \tilde{V})$ of system (2), $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ and $c$ replaced by $\tilde{c} = 1$, given by $\tilde{v} = v = (1, 0, 0)$, $\tilde{h} = h = 0$ and $\tilde{V} = (0, 0, 1)$, which satisfies conditions (4) with $\hat{\epsilon} = 1$ and $\tilde{\epsilon} = -1$.

The corresponding solution $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{N}, \tilde{f})$ of system (27), with $\epsilon = \tilde{\epsilon} = 1$, $c = \tilde{c} = 1$ and initial conditions

$$(\tilde{X}_1(0), \tilde{X}_2(0), \tilde{X}_3(0), \tilde{N}(0), \tilde{f}(0)) = (E_1, E_2, E_3, E_4, E_5)$$

is given by

$$\tilde{f} = \tilde{f}(u_1) = \cos u_1 E_5 + \sin u_1 E_1, \tag{44}$$

$$\tilde{X}_1 = -\sin u_1 E_5 + \cos u_1 E_1, \quad \tilde{X}_2 = E_2, \tag{45}$$

$$\tilde{X}_3 = \cos u_3 E_3 + \sin u_3 E_4 \quad \text{and} \quad \tilde{N} = -\sin u_3 E_3 + \cos u_3 E_4. \tag{46}$$

Arguing as before, we solve system (9) system together with Eqs. (11) and (19), which now become

$$2\tilde{\varphi}\tilde{\psi} = \sum_i \tilde{\gamma}_i^2 + \tilde{\beta}^2 + \tilde{\varphi}^2 \tag{47}$$

and

$$\tilde{v}'^2_2 = \tilde{v}'^2_1 + \tilde{v}'^2_3 - 1. \tag{48}$$

We also impose that

$$\tilde{\varphi}\tilde{v}_3' = \tilde{\beta}(1 - \tilde{v}_1'), \tag{49}$$

which corresponds to the function $\Omega$ in (22) vanishing everywhere. We obtain

$$\tilde{\psi} = \frac{\tilde{\phi}_1^2 - \tilde{\phi}_2^2 + \tilde{\phi}_3^2}{2\tilde{\phi}_1}, \quad (\delta_{i1} - \tilde{v}_i')\tilde{\psi} = \tilde{\phi}_i, \tag{50}$$

$$\tilde{\beta} = \frac{1}{\tilde{K}}\tilde{\phi}_3, \quad \tilde{\varphi} = -\frac{1}{\tilde{K}}\tilde{\phi}_1, \tag{51}$$

$$\tilde{\gamma}_1 = -\frac{1}{\tilde{K}}\tilde{\phi}_1', \quad \tilde{\gamma}_2 = \frac{1}{\tilde{K}}\tilde{\phi}_2' \quad \text{and} \quad \tilde{\gamma}_3 = -\frac{1}{\tilde{K}}\tilde{\phi}_3' \tag{52}$$

for some $\tilde{K} \in \mathbb{R}$, where the functions $\tilde{\phi}_i = \tilde{\phi}_i(u_i)$ satisfy

$$\tilde{\phi}_1'' = -(1 + \tilde{K})\tilde{\phi}_1, \quad \tilde{\phi}_2'' = \tilde{K}\tilde{\phi}_2 \quad \tilde{\phi}_3'' = -(1 + \tilde{K})\tilde{\phi}_3 \tag{53}$$

and

$$(\tilde{\phi}_1'^2 + (1 + \tilde{K})\tilde{\phi}_1^2) + (\phi_2'^2 - \tilde{K}\tilde{\phi}_2^2) + (\phi_3'^2 + (1 + \tilde{K})\tilde{\phi}_3^2) = 0. \tag{54}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (53). Notice also that, for $\tilde{K} = -2$, the two preceding equations coincide with (39), (40) and (41) for $1 = K = a = \epsilon$ and $c = 0$, hence $f'$ and $\tilde{f}'\colon U \to \mathbb{S}^4 \subset \mathbb{R}^5$ share the same induced metric

$$ds^2 = \sum_{i=1}^{3} v_i'^2 du_i^2$$

by (35) and the second equation in (50), where the functions $v_i' = \tilde{v}_i'$, $1 \le i \le 3$, are given explicitly by

$$\begin{aligned} v_1'^2 &= \rho(-2\cos^2\theta\cosh^2 u_1 + 2\sin^2\theta\cosh^2 u_3 - \sin^2\sqrt{2}u_2),\\ v_2'^2 &= 8\rho(\cos^2\theta\cosh^2 u_1 \sin^2\sqrt{2}u_2),\\ v_3'^2 &= 4\sin^2 2\theta\cosh^2 u_1\cosh^2 u_3, \end{aligned}$$

with

$$\rho^{-1} = 2\cos^2\theta\cosh^2 u_1 + 2\sin^2\theta\cosh^2 u_3 - \sin^2\sqrt{2}u_2.$$

The coordinate functions of $\tilde{f}'$ are

$$\begin{aligned} \tilde{f}_1' &= \sin u_1 - gh\cos\theta(\cos u_1\sinh u_1 + \sin u_1\cosh u_1)\\ \tilde{f}_2' &= gh\cos\sqrt{2}u_2\\ \tilde{f}_3' &= -gh\sin\theta(\sin u_3\cosh u_3 + \cos u_3\sinh u_3)\\ \tilde{f}_4' &= gh\sin\theta(\cos u_3\cosh u_3 - \sin u_3\sinh u_3)\\ \tilde{f}_5' &= \cos u_1 + gh\cos\theta(\sin u_1\sinh u_1 - \cos u_1\cosh u_1) \end{aligned} \tag{55}$$

with $g$ and $h$ given by (42) and (43), respectively.

### 4.2 Examples of Conformally Flat Hypersurfaces

One can also use Theorem 9 to compute explicit examples of conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures. It suffices to consider the case $c = 0$, because any conformally flat hypersurface $f\colon M^3 \to \mathbb{Q}_s^4(c)$, $c \neq 0$, is the composition of a conformally flat hypersurface $g\colon M^3 \to \mathbb{R}_s^4$ with an "inverse stereographic projection".

We start with the trivial solution $v = (0, 1, 1)$, $V = (1, 0, 0)$ and $h = 0$ of system (2), for which the corresponding solution of system (27) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, E_4, 0)$$

is given by

$$\begin{aligned} &f = f(u_2, u_3) = u_2E_2 + u_3E_3, \quad X_2 = E_2, \quad X_3 = E_3,\\ &X_1 = \begin{cases} \cosh u_1 E_1 + \sinh u_1 E_4, & \text{if } \epsilon_s = -1,\\ \cos u_1 E_1 + \sin u_1 E_4, & \text{if } \epsilon_s = 1, \end{cases} \end{aligned} \tag{56}$$

and

$$N = \begin{cases} \sinh u_1 E_1 + \cosh u_1 E_4, & \text{if } \epsilon_s = -1, \\ -\sin u_1 E_1 + \cos u_1 E_4, & \text{if } \epsilon_s = 1. \end{cases} \tag{57}$$

Even though this solution does not correspond to a three-dimensional hypersurface, one can still apply Theorem 9. We solve system (9) for $(v, h, V)$ as in the preceding paragraph. Equations (11) and (19), with $K = 0$ in both equations, become

$$2\varphi\psi = \sum_i \gamma_i^2 + \epsilon_s \beta^2 \tag{58}$$

and

$$v_2'^2 = v_1'^2 + v_3'^2. \tag{59}$$

We also impose that

$$\varphi v_1' = -\epsilon_s \beta \left(v_3' - v_2'\right), \tag{60}$$

which corresponds to the function $\Omega$ in (22) vanishing everywhere. It follows from (iii) that

$$v_1'\psi = \beta - \frac{\partial \gamma_1}{\partial u_1}.$$

Since the right-hand-side of the preceding equation depends only on $u_1$ by (ii) and (iv), there exists a smooth function $\phi_1 = \phi_1(u_1)$ such that

$$v_1'\psi = \phi_1. \tag{61}$$

Similarly,

$$(1 - v_i')\psi = \phi_i \tag{62}$$

for some smooth functions $\phi_i = \phi_i(u_i)$, $2 \le i \le 3$. In particular,

$$(v_2' - v_3')\psi = \phi_3 - \phi_2. \tag{63}$$

Multiplying (60) by $\psi$ and using (61) and (63) yields

$$\varphi = \frac{1}{K}(\phi_3 - \phi_2) \tag{64}$$

and

$$\beta = \frac{\epsilon_s}{K}\phi_1 \tag{65}$$

for some $K \in \mathbb{R}$. On the other hand, replacing (61) and (62) in (59), and using (63), we obtain

$$\psi = \frac{\phi_1^2 - \phi_2^2 + \phi_3^2}{2(\phi_3 - \phi_2)}. \tag{66}$$

It follows from (i) and (64) that

$$\gamma_2 = -\frac{1}{K}\phi_2' \quad \text{and} \quad \gamma_3 = \frac{1}{K}\phi_3',$$

whereas (iv) and (65) yield

$$\gamma_1 = -\frac{1}{K}\phi_1'. \tag{67}$$

We obtain from (iii), (61), (65) and (67) that

$$\phi_1'' = (K - \epsilon_s)\phi_1. \tag{68}$$

Similarly,

$$\phi_2'' = -K\phi_2 \quad \text{and} \quad \phi_3'' = K\phi_3. \tag{69}$$

Moreover, by (58) we must have

$$(\phi_1'^2 - (K - \epsilon_s)\phi_1^2) + (\phi_2'^2 + K\phi_2^2) + (\phi_3'^2 - K\phi_3^2) = 0. \tag{70}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (68) and (69).

The conformally flat hypersurface given by (13) (with $c = 0$) has coordinate functions

$$\begin{aligned} f_1' &= (K\psi)^{-1}(\phi_1' \cos u_1 + \phi_1 \sin u_1), \quad f_2' = u_2 + (K\psi)^{-1}\phi_2', \\ f_3' &= u_3 - (K\psi)^{-1}\phi_3' \quad \text{and} \quad f_4' = (K\psi)^{-1}(\phi_1' \sin u_1 - \phi_1 \cos u_1), \end{aligned}$$

if $\epsilon_s = 1$, and

$$\begin{aligned} f_1' &= (K\psi)^{-1}(\phi_1' \cosh u_1 + \phi_1 \sinh u_1), \quad f_2' = u_2 + (K\psi)^{-1}\phi_2', \\ f_3' &= u_3 - (K\psi)^{-1}\phi_3' \quad \text{and} \quad f_4' = (K\psi)^{-1}(\phi_1' \sinh u_1 + \phi_1 \cosh u_1), \end{aligned}$$

if $\epsilon_s = -1$, with $\psi$ as in (66). We compute them explicitly for the particular case $\epsilon_s = 1$ and $K < 0$, the others being similar. In this case we have

$$\begin{cases} \phi_1 = A_{11} \cos\sqrt{|K-1|}\, u_1 + A_{12} \sin\sqrt{|K-1|}\, u_1, \\ \phi_2 = A_{21} \cosh\sqrt{|K|}\, u_2 + A_{22} \sinh\sqrt{|K|}\, u_2, \\ \phi_3 = A_{31} \cos\sqrt{|K|}\, u_3 + A_{32} \sin\sqrt{|K|}\, u_3, \end{cases}$$

with $A_{ij} \in \mathbb{R}$ for $1 \le i, j \le 3$, and Eq. (70) reduces to

$$|K-1|(A_{11}^2 + A_{12}^2) + |K|(A_{22}^2 - A_{21}^2) + |K|(A_{31}^2 + A_{32}^2) = 0.$$

This implies that

$$A_{22}^2 - A_{21}^2 < 0,$$

hence we may write $A_{21} = \rho_2 \cosh\theta_2$ and $A_{22} = \rho_2 \sinh\theta_2$ for some $\rho_2 > 0$ and $\theta_2 \in \mathbb{R}$. We may also write $A_{11} = \rho_1 \cos\theta_1$, $A_{12} = \rho_1 \sin\theta_1$, $A_{31} = \rho_3 \cos\theta_3$ and $A_{32} = \rho_3 \sin\theta_3$ for some $\rho_1, \rho_3 > 0$ and $\theta_1, \theta_3 \in [0, 2\pi]$. Then

$$\begin{cases} \phi_1 = \rho_1 \cos(\sqrt{|K-1|}\, u_1 - \theta_1), \\ \phi_2 = \rho_2 \cosh(\sqrt{|K|}\, u_2 + \theta_2), \\ \phi_3 = \rho_3 \cos(\sqrt{|K|}\, u_3 - \theta_3), \end{cases}$$

with

$$|K|\rho_2^2 = |K-1|\rho_1^2 + |K|\rho_3^2,$$

and we can assume that $\theta_i = 0$ after a suitable change $u_i \mapsto u_i + u_i^0$ of the coordinates $u_i$, $1 \le i \le 3$. Setting $\rho = \rho_2$, we can write $\rho_1 = \sqrt{\frac{|K|}{|K-1|}}\rho \cos\theta$ and $\rho_3 = \rho \sin\theta$ for some $\theta \in [0, 2\pi]$. Thus

$$\begin{cases} \phi_1 = \sqrt{\frac{|K|}{|K-1|}}\rho \cos\theta \cos(\sqrt{|K-1|}\, u_1), \\ \phi_2 = \rho \cosh(\sqrt{|K|}\, u_2), \\ \phi_3 = \rho \sin\theta \cos(\sqrt{|K|}\, u_3). \end{cases}$$

For instance, for $K = -1$ we obtain the one-parameter family (with $\theta$ as the parameter) of conformally flat hypersurfaces of $\mathbb{R}^4$ whose coordinate functions are given by

$$\begin{aligned} f_1' &= 2\cos\theta(\sqrt{2}\cos\sqrt{2}u_1 \sin u_1 - 2\sin\sqrt{2}u_1 \cos u_1)gh, \\ f_2' &= u_2 + 4\sinh u_2 gh, \quad f_3' = u_3 + 4\sin\theta \sin u_3 gh \end{aligned}$$

and

$$f_4' = -2\cos\theta(2\sin\sqrt{2}u_1 \sin u_1 + \sqrt{2}\cos\sqrt{2}u_1 \cos u_1)gh$$

where

$$g = \cosh u_2 - \sin\theta \cos u_3$$

and

$$h^{-1} = \cos^2\theta \cos^2\sqrt{2}u_1 - 2\cosh^2 u_2 + 2\sin^2\theta \cos^2 u_3.$$

One can verify by direct computations that $(u_1, u_2, u_3)$ are principal coordinates for the above hypersurfaces, and that their associated pairs $(v, V)$ satisfy conditions (3). Therefore, they are indeed conformally flat hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures.

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